MTH 101 LECTURE SERIES

Topic: Complex Numbers

By:

Dr. O. J. OMIDIRE

Department of Mathematical Sciences, Osun State University, Osogbo.

February 5, 2026

Course Outline

Contents:

- Complex Numbers (Introduction);
- Algebra of Complex Numbers;
- The Argand Diagram;
- De Moivre's Theorem and Nth Roots of Unity.

Objective

At the end of these lessons, you should be able to:

- define Complex Numbers (in your own words);
- perform some algebraic operations on complex numbers;
- represent complex number diagrammatically on a 2-dimensional plane (Argand diagram); and
- state De Moivre's theorem and express the root of a given complex number in polar.

References

You can consult:

- Essential Mathematics for University Students;
- Engineering Mathematics, by K. A. Stroud; Ross, S. M. (2010). *A First Course in Probability* (8th ed.). Pearson.

Complex Numbers; Introduction and Meaning

Complex numbers, expressed as $a + bj$, where a and b are real numbers and j is the imaginary unit ($j^2 = -1$), are vital in mathematics and physics. They extend the real number system, enabling solutions to equations with no real roots. Complex numbers find applications in diverse fields, from electrical engineering to geometry, owing to their algebraic properties and geometric interpretation on the complex plane.

Introduction

Real Numbers:Real numbers are denoted by $\mathbb{R}$ and they are unions of all types of numbers you are farmiliar with.

Complex Numbers: Complex numbers are numbers that can be expressed in the form $a + bj$, where a and b are real numbers, and j is the imaginary unit, defined as $j^2 = -1$. Complex numbers consist of a real part (a) and an imaginary part (bj).

The Symbol j

The solution of the quadratic equation $m^2 + 1 = 0$ are $m = +\sqrt{-1}, or - \sqrt{-1}$ or simply $\pm\sqrt{-1}$ which can be written as $m = \pm j$. The roots belong to the class of complex numbers, j is called the imaginary part of the solution.In this solution, the real part is 0 since the roots can be written as $m = 0 \pm j$

Powers of j

From the definition of a complex number, j represents $\sqrt{-1}$ and as such

$$j^2 = (\sqrt{-1})^2 = -1$$
$$j^3 = (j)^2 j = -1(j) = -j$$
$$j^4 = (j^2)^2 = (-1)^2 = 1$$

For

$$(j)^8 = (j^4)^2 = (1)^2 = 1$$
$$(j)^9 = (j^4)^2 \cdot j = 1 \cdot j = j$$

Also $(j)^{20} = (j^4)^5 = (1)^5$

Negative Integer Powers of j

Now lets try for negative numbers; negative indices are in the reciprocal of j.
For $j^2 = -1$. dividing through by j

$$\frac{j^2}{j} = \frac{-1}{j}$$
$$j = -j^{-1}$$
$$\implies j^{-1} = -j$$

$$\text{For } j^{-2} = (j^2)^{-1} = (-1)^{-1} = -1$$
$$j^{-3} = (j^{-2}) \cdot j^{-1} = -1 \times -j = j$$
$$j^{-15} = (j^{-3})^5 = (j^5) = (j^4) \cdot j = j$$

Algebral of Complex Numbers

The symbol z is used to represent a complex number. Thus,

$$z = a + jb \tag{1}$$

where a is the real part and b is the imaginary part or $Re(z) = a$ and $Im(z) = jb$. thus

$$Z = Re(z) + Im(z) \tag{2}$$

Algebra of Complex Numbers

Addition and Subtraction. Given that:

$z_1 = a_1 + jb_1$

$z_2 = a_2 + jb_2$

$\vdots$

$z_r = a_r + jb_r$

Then

$z_1 + z_2 + z_3 + z_4 + \cdots + z_r =$

$(a_1 + a_2 + a_3 + a_4 + \cdots + a_r) + j(b_1 + b_2 + b_3 + b_4 + \cdots + b_r)$

$$\implies \sum_{i=1}^{r} z_i = \sum_{i=1}^{r} a_i + j\sum_{i=1}^{r} b_i \tag{3}$$

or

$$\sum_{i=1}^{r} Re_i(z_i) + \sum_{i=1}^{r} Im(z_i) \tag{4}$$

Algebra of Complex Numbers

for Subtraction

$z_1 - z_2 - z_3 - z_4 - \cdots - z_r =$
$(a_1 - a_2 - a_3 - a_4 - \cdots - a_r) - j(b_1 - b_2 - b_3 - b_4 - \cdot - b_r)$

Example 1.1

1. Given that $z_1 = 5 - 3j, z_2 = -3 - 2j, z_3 = 8 - \frac{4}{3}j$ then obtain

(i) $z_1 + z_2 + z_3$

(ii) $z_1 - z_2 - z_3$

Algebra of Complex Numbers

Solution

i. $z_1 + z_2 + z_3 = (5 - 3j) + (-3 - 2j) + (8 - \frac{4}{3}j) =$
$(5 - 3 + 8) + j(-3 - 2 - \frac{4}{3}) = 10 - \frac{19}{3}j$

ii. $z_1 - z_2 - z_3 = (5 - 3j) - (-3 - 2j) - (8 - \frac{4}{3}j) =$
$(5 - 3 - 8) + j(-3 + 2 + \frac{4}{3}) = \frac{1}{3}j$

Multiplication

Multiplication of complex numbers are evaluated just in the same manner we calculate an algebraic expression

Example 1.2

Simplify the following expressions

i. $(2 - 5j)(4j - 5)$

ii. $(7 - 4j)(4 - 3j)$

iii $(6 + 2j)(5 - 3j)$

Algebra of Complex Numbers

Solutions

(i.) $(2-5j)(4j-5) = 2(4j-5) - 5j(4j-5) = 8j - 10 - 20j^2 + 25j$

Recall that $j^2 = -1$

$(2-5j)(4j-5) \implies 8j - 10 - 20(-1) + 25j = 33j + 10$ or $10 + 33j$

(ii.) $(7-4j)(4-3j) = 7(4-3j) - 4j(4-3j) = 28 - 21j - 16j + 12j^2 = 16 - 37j$

(iii.) $(6+2j)(5-3j) = 6(5-3j) + 2j(5-3j) = 36 - 8j$

Complex Conjugate

Complex numbers and their corresponding conjugates are identical only for the sign of the imaginary term. For example, $a + bj$ and $a - bj$ are conjugate complex numbers. However, $a + bj$ and $b - aj$ are not conjugates rather they are distinct complex numbers. Also, $a - bj$ and $-a + jb$ are also not the same complex numbers.

Division of Complex Numbers

For operation of division with complex numbers, the conjugate of the denominating complex number is required

Example 1.3

Simplify the following. $(i)\frac{8-5j}{2}$ $(ii)\frac{2}{8-5j}$

Algebra of Complex Numbers

<u>Solution</u>

(i.) $\frac{8-5j}{2}$ is just a real division and it does not have a complex denominator. therefore the operation is performed just as normal rational

$\frac{8-5j}{2} = \frac{8}{2} - \frac{5j}{2} = 4 - \frac{5j}{2}$

Algebra of Complex Numbers

(ii.) $\frac{2}{8-5j}$ here the denominator is a complex number and as such requires that the conjugate of the denominator is used for its evaluation.

$$\frac{2}{8-5j} = \frac{2}{8-5j} \times \frac{8+5j}{8+5j} = \frac{2(8+5j)}{(8-5j)(8+5j)} = \frac{16+10j}{64+25} = \frac{16+10j}{89} = \frac{16}{89} + \frac{10}{89}j$$

Algebra of Complex Numbers

Equal Complex Numbers Complex numbers are equal only if their real parts are equal and so are the imaginary parts.

Example 1.4: Find the value of m and n if $z_1 = z_2$ where
$z_1 = 5 + jb$ and $z_2 = a + 8j$
Solution
Since $z_1 = z_2$ this implies that $a = 5$ and $b = 8$

Argand Diagram

In 1806, the French mathematician Jean-Robert Argand represented a complex number on a 2-dimensional plane which popularly referred to as Argand Diagram.

Argand diagram is a 2-dimensional plane for graphical representation of a complex number.

Argand Diagram

Example 1.5: Draw an Argand diagram to represent the following complex numbers

(i.) $z_1 = 3 + j5$

(ii.) $z_2 = -4 - 3j$

Solution

Polar Form of a Complex Number

Consider the Argand diagram for $z = x + iy$
Then,
$r^2 = x^2 + y^2$ or $r = \sqrt{x^2 + y^2}$
$\tan\theta = \frac{y}{x}$ or $\tan^{-1}\frac{y}{x} = \theta$
$x = r\cos\theta$ and $y = r\sin\theta$
Since $z = x + jy$, it may be re-written as

$$z = r\cos\theta + jr\sin\theta \quad = r(\cos\theta + j\sin\theta) \tag{5}$$

This is called the polar form of a complex number z.

Polar Form of a Complex Number

Remarks

(1) r is called the modulus of z, denoted by "mod z" or $|z|$

(ii) θ is called the argument of z denoted by "arg z"

Polar Form of a Complex Number

Example 1.6: Express the following complex numbers in polar form.

1. $z_1 = 1 + j1$
2. $z_2 = \sqrt{3} + j$

Polar Form of a Complex Number

<u>Solution.</u>

(1) Let $z_1 = 1 + j$, since $r = \sqrt{(x^2 + y^2)}$, this implies $r = \sqrt{1+1} = \sqrt{2}$.

It is known that $\theta = tan^{-1}(\frac{y}{x})$, this implies $\theta = tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$.

Since $x = r\cos\theta$, therefore, $x = \sqrt{2}\cos\frac{\pi}{4}$,

Similarly, $y = r\sin\theta$, therefore, $y = \sqrt{2}\sin\frac{\pi}{4}$,

since $z = r(\cos\theta + j\sin\theta)$ in polar form, it follows that

$z = \sqrt{2}(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4})$ is the polar form of $z_1 = 1 + j1$

Polar Form of a Complex Number

(2) Let $z_1 = \sqrt{3} + j$, since $r = \sqrt{(}x^2 + y^2)$, this implies $r = \sqrt{(\sqrt{3})^2 + 1} = \sqrt{4} = 2$.
We know that $\theta = tan^{-1}(\frac{y}{x})$, this implies $\theta = tan^{-1}(\frac{1}{\sqrt{(3)}}) = \frac{\pi}{6}$.
since $x = r\cos\theta$, therefore, $x = 2\cos\frac{\pi}{3}$,

Polar Form of a Complex Number

Similarly, $y = r\sin\theta$, therefore, $y = 2\sin\frac{\pi}{3}$,
since $z = r(\cos\theta + j\sin\theta)$ is polar, it follows that
$z = 2(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3})$ is the polar form of $z_1 = \sqrt{3} + j1$

Polar Form of a Complex Number

NOTE: Complex numbers in the form:

(i) $z = x + jy$ has its expression in standard form.

(ii) $z = rL\theta$ or $r\cos\theta + jr\sin\theta$ has its expression in the polar form

(iii) $z = r.e^{j\theta}$ has its expression in exponential form

Roots of a Complex Number

Recall that for a complex number in polar form it can be expressed as follows:

$$z_1 = r_1(\cos\theta_1 + j\sin\theta_1) \tag{6}$$

Also, from Trigonometry Identity, we have:

- $CosACosB + SinASinB = Cos(A - B)$
- $CosAcosB - SinASinB = Cos(A + B)$
- $SinACosB + CosAsinB = Sin(A + B)$

De Moivre's Thorem; Nth Roots of a Complex Number

$$z_2 = r_2(\cos\theta_2 + j\sin\theta_2) \tag{7}$$

then let $z = z_1 z_2$,

$z = r_1 r_2(\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$

$= \mathrm{r}_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1) \right]$

$= \mathrm{r}_1 r_2 \left[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2) \right]$

Roots of a Complex Number

Now, assume that z_1 and z_2 are alike and each is
$z = r(\cos\theta + j\sin\theta)$, $r_1 = r_2, \theta_1 = \theta_2$;
$z^2 = r^2(\cos 2\theta + j\sin 2\theta)$
Thus in general:

Roots of a Complex Number

$$z^n = r^n(\cos\theta + j\sin\theta) \tag{8}$$

Recall that $z = r(\cos\theta + j\sin\theta)$, substituting
$[r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta)$
This is referred to as the De moivre's theorem.

Roots of a Complex Number

Example: 1.7

Simplify the following

(a) $[9(\cos 70^0 + j\sin 70^0)]^2$

(b) $[3(\cos 21^0 + j\sin 21^0)]^3$

(c) $[81(\cos 88^0 + j\sin 88^0)]^{\frac{1}{2}}$

Roots of a Complex Number

<u>Solution</u>

By using the De moivre's theorem

(a) $[9(\cos 70^o + j\sin 70^o)]^2 = 9^2(\cos 2(70^o) + j\sin 2(70^o)) = 81(\cos 140^o + j\sin 140^o)$

(b) $[3(\cos 21^o + j\sin 21^o)]^3 = 3^3(\cos 3(21^o) + j\sin 3(21^o)) = 27(\cos 63^o + j\sin 63^o)$

(c) $[81(\cos 88^o + j\sin 88^o)]^{\frac{1}{2}} = 81^{\frac{1}{2}}(\cos \frac{1}{2}(88^o) + j\sin \frac{1}{2}(88^o)) = 9(\cos 44^o + j\sin 44^o)$

Exercises

(1) Solve the equation $z^3 = 8j$ for z.
(2) Let z be a complex number such that $|z| = 5$ and $\arg(z) = \frac{\pi}{3}$. Find z.
(3) Given that $z = 3 + 4j$ and $w = 5 - 2j$, find $|z + w|^2$.
(4) If $z = 2 + 3j$ and $w = -1 + 2j$, find $\frac{z}{w} + \frac{w}{z}$.
(5) Given that $z = 1 + j$, find $\arg(z^2)$.

THANK YOU
FOR LISTENING,
AND
GOD BLESS YOU