



CSC 203

Discrete Structures

Complete Course Guide
From Beginner to Advanced

2 Units | C: LH 30

Comprehensive Study Material
With Practice Problems & Solutions

PDF prepared by MURKING 



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Chapter 1: Propositional Logic

Propositional logic forms the foundation of mathematical reasoning and computer science. It provides a formal system for analyzing the truth values of statements and their relationships. Understanding propositional logic is essential for programming, circuit design, database queries, and artificial intelligence.

1.1 Introduction to Propositions

Definition 1.1: Proposition

A **proposition** (or **statement**) is a declarative sentence that is either **true** or **false**, but not both. We denote propositions with lowercase letters: p , q , r , s , etc.

Example 1.1: Identifying Propositions

Propositions (have definite truth values):

- p : "2 + 2 = 4" (True)
- q : "The Earth is flat" (False)
- r : "5 is a prime number" (True)

Not Propositions (no definite truth value):

- "What time is it?" (Question)
- "Close the door" (Command)
- "This statement is false" (Paradox)
- " $x + 3 = 7$ " (Depends on value of x)

★ Key Point Trick: The Proposition Test

To check if a sentence is a proposition, ask yourself: **"Can I assign a definite TRUE or FALSE to this statement without any ambiguity?"**

- If YES → It's a proposition
- If NO → It's not a proposition

Memory Aid: Think "P for Proposition, P for Precise truth value"

1.2 Logical Connectives

Logical connectives allow us to combine simple propositions into compound propositions. Each connective has a specific meaning and truth behavior.

Table 1.1 Logical Connectives Summary

Connective	Symbol	Name	Meaning
NOT	$\neg p$ or $\sim p$	Negation	"not p "
AND	$p \wedge q$	Conjunction	" p and q "
OR	$p \vee q$	Disjunction	" p or q " (inclusive)
XOR	$p \oplus q$	Exclusive OR	" p or q but not both"
IMPLIES	$p \rightarrow q$	Implication	"if p then q "
IFF	$p \leftrightarrow q$	Biconditional	" p if and only if q "

1.2.1 Negation ($\neg p$)

Definition 1.2: Negation

The **negation** of a proposition p , denoted $\neg p$ (or $\sim p$), is the proposition "not p ". The negation reverses the truth value: if p is true, $\neg p$ is false, and vice versa.

Example 1.2: Negation Examples

- p : "It is raining" $\rightarrow \neg p$: "It is not raining"
- q : " $5 > 3$ " $\rightarrow \neg q$: " $5 \leq 3$ "
- r : "All birds can fly" $\rightarrow \neg r$: "Some birds cannot fly" (NOT "No birds can fly")

Common Mistake: Negating Universal Statements

When negating statements with quantifiers like "all" or "every":

- "All A are B" negates to "Some A are not B" (not "No A are B")
- "Some A are B" negates to "No A are B" or "All A are not B"

1.2.2 Conjunction ($p \wedge q$)

Definition 1.3: Conjunction

The **conjunction** of propositions p and q , denoted $p \wedge q$, is true **only when both p and q are true**. It is false otherwise.

Example 1.3: Conjunction

Let p : "It is sunny" and q : "It is warm"

$p \wedge q$: "It is sunny AND it is warm" (Both conditions must hold)

1.2.3 Disjunction ($p \vee q$)

Definition 1.4: Disjunction

The **disjunction** of propositions p and q , denoted $p \vee q$, is true **when at least one of p or q is true** (including when both are true). It is false only when both are false.

★ **Key Point Trick: Inclusive vs Exclusive OR**

In logic and mathematics, OR is **inclusive** by default:

- **Inclusive OR (\vee):** "Coffee or tea?" meaning you can have either or both
- **Exclusive OR (\oplus):** "Coffee or tea?" meaning exactly one, not both

Memory Aid: "Inclusive = I might want both"

1.2.4 Implication ($p \rightarrow q$)

Definition 1.5: Implication (Conditional)

The **implication** $p \rightarrow q$ (read "if p then q ") is false **only when p is true and q is false**. It is true in all other cases.

Example 1.4: Implication

Let p : "It rains" and q : "The ground gets wet"

$p \rightarrow q$: "If it rains, then the ground gets wet"

- If it rains and ground gets wet: Statement is TRUE
- If it rains but ground stays dry: Statement is FALSE (promise broken!)
- If it doesn't rain: Statement is TRUE (promise not tested)

★ Key Point Trick: Understanding Implication

Think of $p \rightarrow q$ as a **promise**: "If p happens, I promise q will happen."

- The promise is broken (FALSE) only when p happens but q doesn't
- If p doesn't happen, the promise wasn't tested, so it's kept (TRUE)

Related Terms:

- p = hypothesis/antecedent
- q = conclusion/consequent
- $\neg q \rightarrow \neg p$ = contrapositive (logically equivalent)
- $q \rightarrow p$ = converse (NOT equivalent)
- $\neg p \rightarrow \neg q$ = inverse (NOT equivalent)

1.2.5 Biconditional ($p \leftrightarrow q$)

Definition 1.6: Biconditional

The **biconditional** $p \leftrightarrow q$ (read " p if and only if q ") is true **when p and q have the same truth value** (both true or both false).

Example 1.5: Biconditional

"A triangle is equilateral if and only if all its angles are 60° "

This means: If equilateral \rightarrow all angles 60° , AND if all angles $60^\circ \rightarrow$ equilateral

1.3 Truth Tables

A truth table systematically displays all possible truth values of a compound proposition based on the truth values of its components.

Definition 1.7: Truth Table

A **truth table** is a tabular representation showing all possible combinations of truth values for the component propositions and the resulting truth value of the compound proposition.

Table 1.2 Basic Truth Tables

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

★ **Key Point Trick: Building Truth Tables**

Step-by-step method:

- 1 Count variables (n) → Create 2^n rows
- 2 List all combinations of T/F for variables
- 3 Build columns for sub-expressions
- 4 Final column = complete expression

Pattern for variables: First variable alternates every 2^{n-1} rows, second every 2^{n-2} , etc.

Example 1.6: Complex Truth Table

Construct a truth table for: $(p \wedge q) \rightarrow (\neg p \vee r)$

p	q	r	$p \wedge q$	$\neg p$	$\neg p \vee r$	$(p \wedge q) \rightarrow (\neg p \vee r)$
T	T	T	T	F	T	T
T	T	F	T	F	F	F
T	F	T	F	F	T	T
T	F	F	F	F	F	T
F	T	T	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

1.4 Logical Equivalences

Definition 1.8: Logical Equivalence

Two propositions P and Q are **logically equivalent**, denoted $P \equiv Q$ (or $P \Leftrightarrow Q$), if they have the same truth value in all possible cases.

Theorem 1.1: Important Logical Equivalences**Table 1.3 Fundamental Logical Equivalences**

<i>Name</i>	<i>Equivalence</i>
<i>Identity Laws</i>	$p \wedge T \equiv p, p \vee F \equiv p$
<i>Domination Laws</i>	$p \vee T \equiv T, p \wedge F \equiv F$
<i>Idempotent Laws</i>	$p \vee p \equiv p, p \wedge p \equiv p$
<i>Double Negation</i>	$\neg(\neg p) \equiv p$
<i>Commutative Laws</i>	$p \vee q \equiv q \vee p, p \wedge q \equiv q \wedge p$
<i>Associative Laws</i>	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
<i>Distributive Laws</i>	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
<i>De Morgan's Laws</i>	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
<i>Absorption Laws</i>	$p \vee (p \wedge q) \equiv p, p \wedge (p \vee q) \equiv p$
<i>Negation Laws</i>	$p \vee \neg p \equiv T, p \wedge \neg p \equiv F$
<i>Implication</i>	$p \rightarrow q \equiv \neg p \vee q$
<i>Contrapositive</i>	$p \rightarrow q \equiv \neg q \rightarrow \neg p$
<i>Biconditional</i>	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

★ Key Point Trick: De Morgan's Laws Memory Aid

"Break the line, change the sign"

- $\neg(p \wedge q)$: Break the NOT line over AND \rightarrow change to OR $\rightarrow \neg p \vee \neg q$
- $\neg(p \vee q)$: Break the NOT line over OR \rightarrow change to AND $\rightarrow \neg p \wedge \neg q$

Verbal version: "The negation of 'and' becomes 'or' of negations"

Example 1.7: Proving Equivalence

Prove: $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Proof:

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) && \text{(Implication equivalence)} && (0) \\ &\equiv \neg(\neg p) \wedge \neg q && \text{(De Morgan's Law)} && (0) \\ &\equiv p \wedge \neg q && \text{(Double Negation)} && (0) \end{aligned}$$

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1.5 Normal Forms

Definition 1.9: Disjunctive Normal Form (DNF)

A compound proposition is in **Disjunctive Normal Form (DNF)** if it is a disjunction (OR) of one or more conjunctions (AND) of literals (variables or their negations).

Definition 1.10: Conjunctive Normal Form (CNF)

A compound proposition is in **Conjunctive Normal Form (CNF)** if it is a conjunction (AND) of one or more disjunctions (OR) of literals.

Example 1.8: Normal Forms

DNF Examples:

- $(p \wedge q) \vee (\neg p \wedge r)$
- $p \vee (q \wedge \neg r)$
- $p \wedge q$ (single conjunction)

CNF Examples:

- $(p \vee q) \wedge (\neg p \vee r)$
- $p \wedge (q \vee \neg r)$
- $p \vee q$ (single disjunction)

★ Key Point Trick: Converting to Normal Forms

To DNF from truth table:

1. Identify rows where output is TRUE
2. For each row, create a conjunction of literals (variable if T, negation if F)
3. OR all these conjunctions together

To CNF from truth table:

1. Identify rows where output is FALSE
2. For each row, create a disjunction (opposite: variable if F, negation if T)
3. AND all these disjunctions together

Chapter 1 Summary

- **Proposition:** A statement with a definite truth value (T or F)
- **Logical Connectives:** \neg (NOT), \wedge (AND), \vee (OR), \rightarrow (IMPLIES), \leftrightarrow (IFF)
- **Truth Tables:** Systematic way to evaluate all possible truth values
- **Logical Equivalence:** Two propositions with identical truth tables
- **De Morgan's Laws:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$ and $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- **Normal Forms:** DNF (OR of ANDs), CNF (AND of ORs)

Practice Problems - Chapter 1**Problem 1:** Construct a truth table for $(p \vee q) \rightarrow (p \wedge r)$ **Solution:**

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \rightarrow (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	F	F	T

Problem 2: Use logical equivalences to prove: $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ **Solution:**

$$\begin{aligned}
 p \rightarrow (q \rightarrow r) &\equiv \neg p \vee (q \rightarrow r) && (0) \\
 &\equiv \neg p \vee (\neg q \vee r) && (0) \\
 &\equiv (\neg p \vee \neg q) \vee r && (0) \\
 &\equiv \neg(p \wedge q) \vee r && (0) \\
 &\equiv (p \wedge q) \rightarrow r && (0)
 \end{aligned}$$

Chapter 2: Predicate Logic

While propositional logic deals with complete statements, predicate logic (also called first-order logic) allows us to analyze the internal structure of statements. It introduces variables, predicates, and quantifiers, enabling us to express more complex mathematical statements.

2.1 Predicates and Quantifiers

Definition 2.1: Predicate

A **predicate** is a statement containing variables that becomes a proposition when specific values are substituted for the variables. We denote predicates as $P(x)$, $Q(x, y)$, etc.

Example 2.1: Predicates

- $P(x)$: " $x > 5$ " — Not a proposition until x is specified
- $Q(x, y)$: " $x + y = 10$ " — Depends on values of x and y
- $R(x)$: " x is a prime number" — Truth depends on x

When $x = 7$: $P(7)$ is " $7 > 5$ " which is TRUE

When $x = 3$: $P(3)$ is " $3 > 5$ " which is FALSE

2.1.1 Universal Quantifier (\forall)

Definition 2.2: Universal Quantifier

The **universal quantifier** \forall (read "for all") asserts that a predicate is true for **every** element in a domain. The statement $\forall x P(x)$ means "for all x in the domain, $P(x)$ is true."

Example 2.2: Universal Quantifier

- $\forall x \in \mathbb{R} : x^2 \geq 0$ (True: every real number squared is non-negative)
- $\forall x \in \mathbb{Z} : x > 0$ (False: counterexample $x = -1$)
- $\forall n \in \mathbb{N} : n + 1 > n$ (True)

2.1.2 Existential Quantifier (\exists)**Definition 2.3: Existential Quantifier**

The **existential quantifier** \exists (read "there exists") asserts that a predicate is true for **at least one** element in a domain. The statement $\exists x P(x)$ means "there exists an x in the domain such that $P(x)$ is true."

Example 2.3: Existential Quantifier

- $\exists x \in \mathbb{R} : x^2 = 4$ (True: $x = 2$ or $x = -2$)
- $\exists x \in \mathbb{Z} : x < 0$ (True: e.g., $x = -5$)
- $\exists x \in \mathbb{R} : x^2 = -1$ (False in real numbers)

★ Key Point Trick: Quantifier Memory Aid**"A for All, E for Exists"**

- \forall looks like an upside-down A \rightarrow "All"
- \exists looks like a backward E \rightarrow "Exists"

Truth Conditions:

- $\forall x P(x)$ is TRUE \rightarrow Must check ALL elements (hard to prove)
- $\forall x P(x)$ is FALSE \rightarrow Just need ONE counterexample (easy!)
- $\exists x P(x)$ is TRUE \rightarrow Just need ONE example (easy!)
- $\exists x P(x)$ is FALSE \rightarrow Must check ALL elements (hard!)

2.1.3 Negating Quantified Statements

Theorem 2.1: Negation Rules for Quantifiers

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

Example 2.4: Negating Quantifiers

Original: "All students passed the exam" ($\forall x P(x)$)

Negation: "There exists a student who did not pass" ($\exists x \neg P(x)$)

NOT: "No students passed the exam" (wrong!)

Original: "Some birds cannot fly" ($\exists x \neg F(x)$)

Negation: "All birds can fly" ($\forall x F(x)$)

Common Mistake: Negating Quantifiers

When negating quantified statements:

- Flip \forall to \exists (or vice versa)
- Negate the predicate
- Keep the domain the same

Wrong: $\neg(\forall x P(x)) \equiv \forall x \neg P(x)$ (forgot to flip!)

Right: $\neg(\forall x P(x)) \equiv \exists x \neg P(x)$

2.2 Nested Quantifiers

When multiple quantifiers appear together, their order matters significantly.

Definition 2.4: Nested Quantifiers

Nested quantifiers occur when one quantifier appears within the scope of another. The order of quantifiers affects the meaning of the statement.

Table 2.1 Nested Quantifier Examples

Statement	Meaning
$\forall x \forall y P(x, y)$	For all x and for all y , $P(x, y)$ holds
$\exists x \exists y P(x, y)$	There exists an x and there exists a y such that $P(x, y)$ holds
$\forall x \exists y P(x, y)$	For every x , there exists a y (may depend on x) such that $P(x, y)$
$\exists x \forall y P(x, y)$	There exists an x (fixed) such that for all y , $P(x, y)$ holds

Example 2.5: Order Matters!

Let $L(x, y)$: " x loves y " where domain is all people.

- $\forall x \exists y L(x, y)$: "Everyone loves someone" (each person has someone they love)
- $\exists y \forall x L(x, y)$: "There is someone whom everyone loves" (a universally loved person)

These are **not equivalent!** The second is much stronger than the first.

★ **Key Point Trick: Reading Nested Quantifiers**

Method: Read from left to right, thinking of quantifiers as nested loops

- $\forall x \exists y$: For each x , I can find a y (possibly different for each x)
- $\exists x \forall y$: First find a special x , then check it works for all y

Memory Rule: $\forall \exists$ is weaker than $\exists \forall$

2.3 Rules of Inference

Rules of inference are valid argument forms that allow us to draw conclusions from premises.

Definition 2.5: Argument

An **argument** in propositional logic is a sequence of propositions where all but the final proposition are called **premises** (or hypotheses), and the final proposition is called the **conclusion**. An argument is **valid** if the conclusion follows logically from the premises.

Table 2.2 Common Rules of Inference

Rule	Tautology	Name
$p \rightarrow q$ p $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus Ponens
$p \rightarrow q$ $\neg q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus Tollens
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive Syllogism
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Simplification
p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	Addition
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical Syllogism

★ **Key Point Trick: Remembering Inference Rules**

Modus Ponens (Affirming): If $p \rightarrow q$ and you have p , you get q

Modus Tollens (Denying): If $p \rightarrow q$ and you have $\neg q$, you get $\neg p$

Memory: "MP = Make Positive, MT = Make True (the contrapositive)"

Example 2.6: Using Rules of Inference

Given:

1. If it rains, the ground gets wet. ($r \rightarrow w$)
2. If the ground gets wet, the grass grows. ($w \rightarrow g$)
3. It is raining. (r)

Prove: The grass grows. (g)***Proof:***

Step	Statement	Justification
1	$r \rightarrow w$	Premise
2	$w \rightarrow g$	Premise
3	r	Premise
4	w	Modus Ponens (1, 3)
5	g	Modus Ponens (2, 4)

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Common Mistake: Fallacies

Affirming the Consequent (INVALID):

$p \rightarrow q, q$, therefore p — WRONG!

Example: "If it's a dog, it's an animal. It's an animal. Therefore it's a dog." (Could be a cat!)

Denying the Antecedent (INVALID):

$p \rightarrow q, \neg p$, therefore $\neg q$ — WRONG!

Example: "If it's a dog, it's an animal. It's not a dog. Therefore it's not an animal." (Could be a cat!)

Chapter 2 Summary

- **Predicate:** Statement with variables $P(x)$ that becomes proposition when values substituted
- **Universal Quantifier (\forall):** "For all" — must be true for every element
- **Existential Quantifier (\exists):** "There exists" — true for at least one element
- **Negation Rules:** $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$
- **Nested Quantifiers:** Order matters! $\forall x\exists y \neq \exists y\forall x$
- **Rules of Inference:** Valid argument forms (Modus Ponens, Modus Tollens, etc.)

Practice Problems - Chapter 2**Problem 1:** Negate: $\forall x \exists y (x + y = 0)$ **Solution:**

$$\neg(\forall x \exists y (x + y = 0)) \equiv \exists x \forall y \neg(x + y = 0) \equiv \exists x \forall y (x + y \neq 0)$$

"There exists an x such that for all y , $x + y \neq 0$ "

Problem 2: Prove using rules of inference:Premises: (1) $p \rightarrow q$, (2) $\neg r \rightarrow \neg q$, (3) p Conclusion: r **Solution:**

Step	Statement	Justification
1	$p \rightarrow q$	Premise
2	$\neg r \rightarrow \neg q$	Premise
3	p	Premise
4	q	Modus Ponens (1, 3)
5	$q \rightarrow r$	Contrapositive of (2)
6	r	Modus Ponens (4, 5)

Chapter 3: Sets

Set theory is the foundation of modern mathematics. A set is simply a collection of distinct objects, and set theory provides the language and notation for describing relationships between collections.

3.1 Set Definitions and Notations

Definition 3.1: Set

A **set** is an unordered collection of distinct objects, called **elements** or **members** of the set. We write $a \in A$ to denote that a is an element of set A , and $a \notin A$ to denote that a is not an element of A .

3.1.1 Methods of Describing Sets

Table 3.1 Set Description Methods

Method	Notation	Example
Roster Method	List elements in braces	$A = \{1, 2, 3, 4, 5\}$
Set-Builder	$\{x \mid P(x)\}$	$B = \{x \in \mathbb{Z} \mid 1 \leq x \leq 5\}$
Interval	$[a, b], (a, b)$	$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

Example 3.1: Set Notations

- $\{1, 2, 3\} = \{x \in \mathbb{Z}^+ \mid x \leq 3\} = \{x \in \mathbb{N} \mid 1 \leq x \leq 3\}$
- $\{2, 4, 6, 8, \dots\} = \{x \in \mathbb{Z}^+ \mid x \text{ is even}\} = \{2n \mid n \in \mathbb{Z}^+\}$
- $\{x \in \mathbb{R} \mid x^2 < 4\} = (-2, 2)$

3.1.2 Important Sets

Table 3.2 Standard Sets in Mathematics

Set	Symbol	Description
Natural Numbers	\mathbb{N}	$\{1, 2, 3, \dots\}$ (or $\{0, 1, 2, \dots\}$)
Integers	\mathbb{Z}	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational Numbers	\mathbb{Q}	$\{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$
Real Numbers	\mathbb{R}	All points on the number line
Complex Numbers	\mathbb{C}	$\{a + bi \mid a, b \in \mathbb{R}\}$
Empty Set	\emptyset or $\{\}$	Set with no elements
Universal Set	U	Set of all elements under consideration

★ Key Point Trick: Remembering Number Sets

Memory Aid: "Nice Zebras Quite Rapidly Climb"

- Natural → Zahlen (German for integers) → Quotient → Real → Complex

Hierarchy: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

3.1.3 Set Relationships

Definition 3.2: Subset

Set A is a **subset** of set B , denoted $A \subseteq B$, if every element of A is also an element of B . Formally: $\forall x(x \in A \rightarrow x \in B)$

Definition 3.3: Proper Subset

Set A is a **proper subset** of set B , denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$ (there exists at least one element in B not in A).

Definition 3.4: Set Equality

Two sets A and B are **equal**, denoted $A = B$, if they contain exactly the same elements. Formally: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Example 3.2: Set Relationships

- $\{1, 2\} \subseteq \{1, 2, 3\}$ (True)
- $\{1, 2\} \subset \{1, 2, 3\}$ (True — proper subset)
- $\{1, 2, 3\} \subseteq \{1, 2, 3\}$ (True — every set is a subset of itself)
- $\{1, 2, 3\} \subset \{1, 2, 3\}$ (False — not proper)
- $\emptyset \subseteq A$ for any set A (True — empty set is subset of every set)

Common Mistake: Element vs Subset

Distinguish between:

- $a \in A$: a is an **element** of A
- $\{a\} \subseteq A$: The set containing a is a **subset** of A

Wrong: $1 \subseteq \{1, 2, 3\}$ (1 is not a set)

Right: $1 \in \{1, 2, 3\}$ or $\{1\} \subseteq \{1, 2, 3\}$

3.2 Set Operations**Definition 3.5: Union**

The **union** of sets A and B , denoted $A \cup B$, is the set of all elements that are in A or in B (or in both).

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Definition 3.6: Intersection

The **intersection** of sets A and B , denoted $A \cap B$, is the set of all elements that are in both A and B .

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Definition 3.7: Set Difference

The **difference** of sets A and B , denoted $A - B$ or $A \setminus B$, is the set of elements in A that are not in B .

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Definition 3.8: Complement

The **complement** of set A , denoted \overline{A} or A^c , is the set of all elements in the universal set U that are not in A .

$$\overline{A} = \{x \in U \mid x \notin A\} = U - A$$

Definition 3.9: Symmetric Difference

The **symmetric difference** of sets A and B , denoted $A \oplus B$, is the set of elements in exactly one of A or B .

$$A \oplus B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Example 3.3: Set Operations

Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

- $A \cup B = \{1, 2, 3, 4, 5, 6\}$
- $A \cap B = \{3, 4\}$
- $A - B = \{1, 2\}$
- $B - A = \{5, 6\}$
- $\bar{A} = \{5, 6, 7, 8\}$
- $A \oplus B = \{1, 2, 5, 6\}$

★ Key Point Trick: Venn Diagram Regions

For two sets A and B in a Venn diagram:

- Only A (not B): $A - B$
- Only B (not A): $B - A$
- Both A and B : $A \cap B$
- Either A or B : $A \cup B$
- Neither A nor B : $\overline{A \cup B}$

Memory: Think of \cup as "U" for Union (everything), \cap as "n" for iNtersection (narrow, only overlap)

3.3 Set Identities

Theorem 3.1: Fundamental Set Identities

Table 3.3 Set Identities (Analogous to Logical Equivalences)

<i>Name</i>	<i>Set Identity</i>	<i>Logic Analog</i>
<i>Identity</i>	$A \cup \emptyset = A, A \cap U = A$	$p \vee F = p, p \wedge T = p$
<i>Domination</i>	$A \cup U = U, A \cap \emptyset = \emptyset$	$p \vee T = T, p \wedge F = F$
<i>Idempotent</i>	$A \cup A = A, A \cap A = A$	$p \vee p = p, p \wedge p = p$
<i>Complementation</i>	$\overline{\overline{A}} = A$	$\neg\neg p = p$
<i>Commutative</i>	$A \cup B = B \cup A, A \cap B = B \cap A$	$p \vee q = q \vee p$
<i>Associative</i>	$(A \cup B) \cup C = A \cup (B \cup C)$	$(p \vee q) \vee r = p \vee (q \vee r)$
<i>Distributive</i>	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
<i>De Morgan's</i>	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\neg(p \vee q) = \neg p \wedge \neg q$
<i>Absorption</i>	$A \cup (A \cap B) = A$	$p \vee (p \wedge q) = p$
<i>Complement</i>	$A \cup \overline{A} = U, A \cap \overline{A} = \emptyset$	$p \vee \neg p = T, p \wedge \neg p = F$

★ Key Point Trick: Set-Logic Correspondence

Sets and logic are deeply connected:

Set Operation	Logical Connective
\cup (Union)	\vee (OR)
\cap (Intersection)	\wedge (AND)
\overline{A} (Complement)	\neg (NOT)
U (Universal)	T (True)
\emptyset (Empty)	F (False)
\subseteq (Subset)	\rightarrow (Implies)

3.4 Power Sets and Cartesian Products

Definition 3.10: Power Set

The **power set** of a set A , denoted $\mathcal{P}(A)$ or 2^A , is the set of all subsets of A .

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}$$

Theorem 3.2: Power Set Cardinality

If a set A has n elements, then its power set $\mathcal{P}(A)$ has 2^n elements.

$$|A| = n \Rightarrow |\mathcal{P}(A)| = 2^n$$

Example 3.4: Power Set

Let $A = \{1, 2, 3\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Number of elements: $2^3 = 8$

Definition 3.11: Cartesian Product

The **Cartesian product** of sets A and B , denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Theorem 3.3: Cartesian Product Cardinality

$$|A \times B| = |A| \cdot |B|$$

For n sets: $|A_1 \times A_2 \times \dots \times A_n| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_n|$

Example 3.5: Cartesian Product

Let $A = \{1, 2\}$ and $B = \{a, b, c\}$

$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

$$|A \times B| = 2 \times 3 = 6$$

★ Key Point Trick: Power Set Pattern

To list all subsets of $\{a, b, c\}$ systematically:

Size 0:	\emptyset
Size 1:	$\{a\}, \{b\}, \{c\}$
Size 2:	$\{a, b\}, \{a, c\}, \{b, c\}$
Size 3:	$\{a, b, c\}$

Binary representation trick: Each subset corresponds to a binary number from 0 to $2^n - 1$

For $\{a, b, c\}$: $000 = \emptyset$, $001 = \{c\}$, $010 = \{b\}$, $011 = \{b, c\}$, etc.

Chapter 3 Summary

- **Set:** Unordered collection of distinct elements
- **Subset (\subseteq):** All elements of A are in B
- **Union (\cup):** Elements in A or B
- **Intersection (\cap):** Elements in both A and B
- **Difference ($-$):** Elements in A but not in B
- **Complement (\bar{A}):** Elements not in A (relative to U)
- **Power Set $\mathcal{P}(A)$:** All subsets of A; $|\mathcal{P}(A)| = 2^{|A|}$
- **Cartesian Product (\times):** Set of ordered pairs; $|A \times B| = |A| \cdot |B|$

Practice Problems - Chapter 3

Problem 1: Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$, $C = \{1, 3, 5\}$. Find $(A \cap B) \cup C$.

Solution:

$$A \cap B = \{2, 4\}$$

$$(A \cap B) \cup C = \{2, 4\} \cup \{1, 3, 5\} = \{1, 2, 3, 4, 5\}$$

Problem 2: Prove using set identities: $A - (B \cap C) = (A - B) \cup (A - C)$

Solution:

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{(B \cap C)} && \text{(Definition of difference)} && (0) \\ &= A \cap (\overline{B} \cup \overline{C}) && \text{(De Morgan's Law)} && (0) \\ &= (A \cap \overline{B}) \cup (A \cap \overline{C}) && \text{(Distributive Law)} && (0) \\ &= (A - B) \cup (A - C) && \text{(Definition of difference)} && (0) \end{aligned}$$

Chapter 4: Functions

Functions are fundamental mathematical objects that describe relationships between sets. In computer science, functions model algorithms, transformations, and mappings between data structures.

4.1 Function Definitions

Definition 4.1: Function

Let A and B be sets. A **function** (or **mapping**) f from A to B , denoted $f : A \rightarrow B$, is an assignment of exactly one element of B to each element of A . We write $f(a) = b$ if b is the element assigned to a .

Definition 4.2: Domain, Codomain, Range

- **Domain:** The set A (input values)
- **Codomain:** The set B (possible output values)
- **Range/Image:** The set $\{f(a) \mid a \in A\}$ (actual output values)

Example 4.1: Function Components

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$

- **Domain:** \mathbb{R} (all real numbers)
- **Codomain:** \mathbb{R} (all real numbers)
- **Range:** $[0, \infty)$ (only non-negative reals)

Note: $\text{Range} \subseteq \text{Codomain}$, but $\text{Range} \neq \text{Codomain}$ in this case.

4.1.1 Representing Functions

Table 4.1 Function Representation Methods

Method	Description	Example
Formula	Algebraic expression	$f(x) = 2x + 1$
Table	List input-output pairs	$\{(1, 3), (2, 5), (3, 7)\}$
Arrow Diagram	Visual mapping	$1 \rightarrow 3, 2 \rightarrow 5$
Set of Ordered Pairs	Subset of $A \times B$	$\{(a, f(a)) \mid a \in A\}$

★ Key Point Trick: Function Test

A relation is a function if and only if:

1. **Every** element in the domain has an output (total)
2. Each element has **exactly one** output (well-defined)

Visual test: Vertical line test for graphs — each vertical line intersects at most once.

4.2 Types of Functions

Definition 4.3: Injective (One-to-One)

A function $f : A \rightarrow B$ is **injective** (or **one-to-one**) if distinct inputs give distinct outputs:

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

Equivalently: $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Definition 4.4: Surjective (Onto)

A function $f : A \rightarrow B$ is **surjective** (or **onto**) if every element in B is mapped to by some element in A :

$$\forall b \in B, \exists a \in A : f(a) = b$$

Equivalently: Range = Codomain

Definition 4.5: Bijective

A function $f : A \rightarrow B$ is **bijective** if it is both injective and surjective. A bijective function establishes a **one-to-one correspondence** between A and B .

Example 4.2: Function Types

Injective but not surjective: $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = e^x$

- Injective: $e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$
- Not surjective: negative numbers not in range

Surjective but not injective: $f : \mathbb{R} \rightarrow [0, \infty), f(x) = x^2$

- Not injective: $f(2) = f(-2) = 4$
- Surjective: every non-negative number is a square

Bijective: $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 1$

★ Key Point Trick: Remembering Function Types

Memory Aid: "ISO"

- Injective = Into (no collisions, fits into codomain)
- Surjective = Spans (covers entire codomain)
- Bijective = Both (perfect matching)

Horizontal Line Test:

- Injective: Each horizontal line hits at most once
- Surjective: Each horizontal line hits at least once
- Bijective: Each horizontal line hits exactly once

Common Mistake: Proving Injectivity

Wrong approach: Show $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ (this is true for ALL functions!)

Right approach: Show $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ (the converse)

Or prove the contrapositive: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

4.3 Function Composition

Definition 4.6: Function Composition

Let $f : A \rightarrow B$ and $g : B \rightarrow C$. The **composition** of g with f , denoted $g \circ f$, is the function from A to C defined by:

$$(g \circ f)(a) = g(f(a))$$

Note: The domain of g must contain the range of f .

Example 4.3: Function Composition

Let $f(x) = x + 1$ and $g(x) = x^2$

- $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2$
- $(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$

Note: $g \circ f \neq f \circ g$ (composition is not commutative!)

Theorem 4.1: Properties of Composition

1. Composition is associative: $(h \circ g) \circ f = h \circ (g \circ f)$
2. If f and g are injective, then $g \circ f$ is injective
3. If f and g are surjective, then $g \circ f$ is surjective
4. If f and g are bijective, then $g \circ f$ is bijective

4.4 Inverse Functions**Definition 4.7: Inverse Function**

Let $f : A \rightarrow B$ be a bijection. The **inverse function** of f , denoted $f^{-1} : B \rightarrow A$, is defined by:

$$f^{-1}(b) = a \text{ if and only if } f(a) = b$$

Theorem 4.2: Existence of Inverse

A function has an inverse if and only if it is **bijective**.

Theorem 4.3: Properties of Inverse

If f is bijective with inverse f^{-1} :

1. $f^{-1} \circ f = \text{id}_A$ (identity on A)
2. $f \circ f^{-1} = \text{id}_B$ (identity on B)
3. $(f^{-1})^{-1} = f$

Example 4.4: Finding Inverse

Find the inverse of $f(x) = 2x + 3$

Solution:

$$\text{Let } y = f(x) = 2x + 3$$

$$\text{Solve for } x: y - 3 = 2x$$

$$x = \frac{y-3}{2}$$

$$\text{Therefore: } f^{-1}(y) = \frac{y-3}{2} \text{ or } f^{-1}(x) = \frac{x-3}{2}$$

$$\text{Verification: } (f^{-1} \circ f)(x) = f^{-1}(2x + 3) = \frac{(2x+3)-3}{2} = x \checkmark$$

★ **Key Point Trick: Finding Inverse Functions**

Step-by-step method:

- 1 Write $y = f(x)$
- 2 Solve for x in terms of y
- 3 Swap x and y (optional, for standard notation)
- 4 Verify: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Common Mistake: Inverse Notation

$f^{-1}(x)$ means the **inverse function**, NOT $\frac{1}{f(x)}$

For example, if $f(x) = x^2$, then $f^{-1}(x) = \sqrt{x}$ (for $x \geq 0$), not $\frac{1}{x^2}$

Chapter 4 Summary

- **Function** $f : A \rightarrow B$: Assigns exactly one output in B to each input in A
- **Injective (One-to-One)**: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
- **Surjective (Onto)**: Range = Codomain
- **Bijjective**: Both injective and surjective (has inverse)
- **Composition** $(g \circ f)(x)$: $g(f(x))$ — apply f first, then g
- **Inverse** f^{-1} : Exists iff f is bijective; undoes f

Practice Problems - Chapter 4

Problem 1: Determine if $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3 - x$ is injective.

Solution:

Check if $f(a) = f(b) \Rightarrow a = b$:

$$a^3 - a = b^3 - b$$

$$a^3 - b^3 = a - b$$

$$(a - b)(a^2 + ab + b^2) = (a - b)$$

Either $a = b$ or $a^2 + ab + b^2 = 1$

Counterexample: $f(0) = 0$ and $f(1) = 0$, but $0 \neq 1$

Not injective.

Problem 2: Find the inverse of $f(x) = \frac{2x-1}{x+3}$

Solution:

$$\text{Let } y = \frac{2x-1}{x+3}$$

$$y(x+3) = 2x-1$$

$$yx + 3y = 2x - 1$$

$$yx - 2x = -1 - 3y$$

$$x(y-2) = -(1+3y)$$

$$x = \frac{-(1+3y)}{y-2} = \frac{1+3y}{2-y}$$

$$f^{-1}(x) = \frac{1+3x}{2-x}$$

Chapter 5: Sequences and Summation

Sequences and summations are fundamental tools in discrete mathematics, appearing in algorithm analysis, combinatorics, and many other areas of computer science.

5.1 Sequences

Definition 5.1: Sequence

A **sequence** is a function from a subset of integers (typically $\{0, 1, 2, \dots\}$ or $\{1, 2, 3, \dots\}$) to a set S . We denote a sequence as $\{a_n\}$ or list its terms: a_1, a_2, a_3, \dots

5.1.1 Types of Sequences

Table 5.1 Common Sequence Types

Type	Definition	Example
Finite	Defined for finite set of indices	1, 2, 3, 4, 5
Infinite	Continues indefinitely	1, 2, 3, 4, 5, ...
Increasing	$a_n < a_{n+1}$ for all n	2, 4, 6, 8, ...
Decreasing	$a_n > a_{n+1}$ for all n	10, 8, 6, 4, ...
Non-decreasing	$a_n \leq a_{n+1}$ for all n	1, 1, 2, 2, 3, ...

Example 5.1: Sequence Examples

- $a_n = n^2$: 1, 4, 9, 16, 25, ...
- $a_n = 2^n$: 2, 4, 8, 16, 32, ...
- $a_n = (-1)^n$: -1, 1, -1, 1, -1, ... (alternating)
- $a_n = \frac{1}{n}$: 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...

5.1.2 Recursive Definitions

Definition 5.2: Recursive Sequence

A sequence is defined **recursively** by specifying:

1. **Base case(s):** Initial term(s)
2. **Recurrence relation:** Rule to compute subsequent terms from previous ones

Example 5.2: Fibonacci Sequence

The Fibonacci sequence is defined by:

- Base: $F_0 = 0, F_1 = 1$
- Recurrence: $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

★ Key Point Trick: Famous Sequences

Sequence	Formula	First Terms
Natural numbers	$a_n = n$	1, 2, 3, 4, 5, ...
Even numbers	$a_n = 2n$	2, 4, 6, 8, 10, ...
Odd numbers	$a_n = 2n - 1$	1, 3, 5, 7, 9, ...
Squares	$a_n = n^2$	1, 4, 9, 16, 25, ...
Cubes	$a_n = n^3$	1, 8, 27, 64, 125, ...
Powers of 2	$a_n = 2^n$	2, 4, 8, 16, 32, ...
Factorials	$a_n = n!$	1, 2, 6, 24, 120, ...

5.2 Summation Notation

Definition 5.3: Summation

The **summation** of terms a_m, a_{m+1}, \dots, a_n is denoted:

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_n$$

where i is the **index of summation**, m is the **lower limit**, and n is the **upper limit**.

Example 5.3: Summation Examples

- $\sum_{i=1}^5 i = 1 + 2 + 3 + 4 + 5 = 15$
- $\sum_{i=0}^3 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15$
- $\sum_{j=2}^4 j^2 = 2^2 + 3^2 + 4^2 = 4 + 9 + 16 = 29$

5.2.1 Properties of Summation

Theorem 5.1: Summation Properties

1. **Linearity:** $\sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i$
2. **Constant multiple:** $\sum_{i=m}^n c \cdot a_i = c \cdot \sum_{i=m}^n a_i$
3. **Constant sum:** $\sum_{i=m}^n c = c \cdot (n - m + 1)$
4. **Index shift:** $\sum_{i=m}^n a_i = \sum_{j=0}^{n-m} a_{j+m}$

5.3 Arithmetic and Geometric Sequences

Definition 5.4: Arithmetic Sequence

An **arithmetic sequence** has a constant difference d between consecutive terms:

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term and d is the **common difference**.

Definition 5.5: Geometric Sequence

A **geometric sequence** has a constant ratio r between consecutive terms:

$$a_n = a_1 \cdot r^{n-1}$$

where a_1 is the first term and r is the **common ratio**.

Example 5.4: Arithmetic and Geometric

Arithmetic: 3, 7, 11, 15, 19, ... ($a_1 = 3, d = 4$)

$$a_n = 3 + (n - 1) \cdot 4 = 4n - 1$$

Geometric: 2, 6, 18, 54, 162, ... ($a_1 = 2, r = 3$)

$$a_n = 2 \cdot 3^{n-1}$$

5.4 Summation Formulas

Theorem 5.2: Important Summation Formulas

Table 5.2 Standard Summation Formulas

Sum	Formula
$\sum_{i=1}^n i$	$\frac{n(n+1)}{2}$
$\sum_{i=1}^n i^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^n i^3$	$\left(\frac{n(n+1)}{2}\right)^2$
$\sum_{i=0}^n r^i \ (r \neq 1)$	$\frac{r^{n+1}-1}{r-1}$
$\sum_{i=0}^{\infty} r^i \ (r < 1)$	$\frac{1}{1-r}$
$\sum_{i=1}^n c$	cn

★ Key Point Trick: Remembering Summation Formulas

Gauss's Formula (sum of first n integers):

Pair first and last: $1 + n = n + 1, 2 + (n - 1) = n + 1$, etc.

Number of pairs = $n/2$, each pair sums to $(n + 1)$

$$\text{Total} = \frac{n(n+1)}{2}$$

Geometric Series: "First term minus what comes next, divided by 1 minus the ratio"

$$S = \frac{a_1 - a_1 r^n}{1-r} = \frac{a_1(1-r^n)}{1-r}$$

Example 5.5: Using Summation Formulas**Problem:** Find $\sum_{i=1}^{100} i$ **Solution:** $\frac{100 \cdot 101}{2} = 5050$ **Problem:** Find $\sum_{i=0}^5 2^i$ **Solution:** $\frac{2^6 - 1}{2 - 1} = \frac{64 - 1}{1} = 63$ **Common Mistake: Summation Index**

Be careful with the starting index!

- $\sum_{i=0}^n i = 0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Both give the same result because adding 0 doesn't change the sum!

But: $\sum_{i=0}^n r^i = 1 + r + r^2 + \dots + r^n$ has $(n + 1)$ terms**Chapter 5 Summary**

- **Sequence:** Ordered list of numbers $\{a_n\}$
- **Arithmetic:** $a_n = a_1 + (n - 1)d$ (constant difference)
- **Geometric:** $a_n = a_1 \cdot r^{n-1}$ (constant ratio)
- **Summation \sum :** Compact notation for adding terms
- **Key Formulas:**
 - $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
 - $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
 - $\sum_{i=0}^n r^i = \frac{r^{n+1} - 1}{r - 1}$

Practice Problems - Chapter 5**Problem 1:** Find $\sum_{i=1}^{20} (2i - 1)$ **Solution:**

$$\sum_{i=1}^{20} (2i - 1) = 2 \sum_{i=1}^{20} i - \sum_{i=1}^{20} 1 = 2 \cdot \frac{20 \cdot 21}{2} - 20 = 420 - 20 = 400$$

(This is the sum of first 20 odd numbers, which equals $20^2 = 400$)**Problem 2:** Find $\sum_{i=0}^{10} \left(\frac{1}{2}\right)^i$ **Solution:**

$$\sum_{i=0}^{10} \left(\frac{1}{2}\right)^i = \frac{1 - (1/2)^{11}}{1 - 1/2} = \frac{1 - 1/2048}{1/2} = 2 \cdot \frac{2047}{2048} = \frac{2047}{1024}$$

Chapter 6: Proof Techniques

Mathematical proofs are the foundation of mathematical reasoning. A proof is a valid argument that establishes the truth of a mathematical statement. This chapter covers the most common proof techniques used in discrete mathematics.

6.1 Direct Proof

Definition 6.1: Direct Proof

A **direct proof** establishes the truth of a conditional statement $p \rightarrow q$ by assuming p is true and then using definitions, axioms, and previously proven theorems to show that q must also be true.

Example 6.1: Direct Proof

Theorem: If n is an even integer, then n^2 is even.

Proof:

Assume n is even. By definition, $n = 2k$ for some integer k .

Then $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.

Since $2k^2$ is an integer, $n^2 = 2m$ where $m = 2k^2$.

Therefore, n^2 is even. ■

★ Key Point Trick: Direct Proof Strategy

For proving $p \rightarrow q$:

- 1 Assume p is true
- 2 Expand definitions of terms in p
- 3 Use logical deductions, algebra, theorems
- 4 Arrive at q

6.2 Proof by Contrapositive

Definition 6.2: Contrapositive

The **contrapositive** of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. A conditional statement is logically equivalent to its contrapositive.

Example 6.2: Proof by Contrapositive

Theorem: If n^2 is even, then n is even.

(Direct proof would be difficult!)

Proof (by contrapositive):

We prove: If n is odd, then n^2 is odd.

Assume n is odd. Then $n = 2k + 1$ for some integer k .

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2m + 1$$

where $m = 2k^2 + 2k$ is an integer.

Therefore, n^2 is odd. ■

★ **Key Point Trick: When to Use Contrapositive**

Use contrapositive when:

- The conclusion (q) gives more useful information than the hypothesis (p)
- Direct proof seems difficult
- Proving "not p " is easier than proving q

Common applications: Proofs involving "not divisible by", "not equal to", etc.

6.3 Proof by Contradiction

Definition 6.3: Proof by Contradiction

To prove statement p by contradiction:

1. Assume $\neg p$ (the negation of p)
2. Show that this assumption leads to a contradiction (a statement that is always false)
3. Conclude that p must be true

Example 6.3: Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof:

Assume $\sqrt{2}$ is rational. Then $\sqrt{2} = p/q$ where p, q are integers with no common factors (fraction in lowest terms).

Squaring: $2 = p^2/q^2$, so $p^2 = 2q^2$.

This means p^2 is even, so p is even (by Example 6.2).

So $p = 2k$ for some integer k .

Then $(2k)^2 = 2q^2$, so $4k^2 = 2q^2$, thus $q^2 = 2k^2$.

This means q^2 is even, so q is even.

But then both p and q are even, contradicting that they have no common factors.

Therefore, $\sqrt{2}$ is irrational. ■

★ **Key Point Trick: Contradiction vs Contrapositive**

	Contrapositive	Contradiction
Goal	Prove $p \rightarrow q$	Prove any statement p
Assume	$\neg q$	$\neg p$
Prove	$\neg p$	Any contradiction
Best for	Conditionals	Existence, irrationality

6.4 Proof by Cases

Definition 6.4: Proof by Cases

To prove a statement by cases:

1. Divide the problem into exhaustive cases that cover all possibilities
2. Prove the statement for each case separately
3. Conclude the statement holds in general

Example 6.4: Proof by Cases

Theorem: For any integer n , $n^2 + n$ is even.

Proof:

Case 1: n is even. Then $n = 2k$.

$$n^2 + n = 4k^2 + 2k = 2(2k^2 + k), \text{ which is even.}$$

Case 2: n is odd. Then $n = 2k + 1$.

$$n^2 + n = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1), \text{ which is even.}$$

In both cases, $n^2 + n$ is even. ■

★ **Key Point Trick: Common Case Divisions**

- **Parity:** Even vs Odd
- **Sign:** Positive, Negative, Zero
- **Divisibility:** Remainders when divided by n
- **Comparisons:** $a < b$, $a = b$, $a > b$

Key: Cases must be exhaustive (cover all possibilities) and ideally mutually exclusive.

Chapter 6 Summary

- **Direct Proof:** Assume p , prove q for $p \rightarrow q$
- **Contrapositive:** Prove $\neg q \rightarrow \neg p$ instead of $p \rightarrow q$
- **Contradiction:** Assume $\neg p$, derive contradiction
- **Cases:** Divide into exhaustive cases, prove each
- **Choose method based on:** Statement structure, available information, what seems easier

Practice Problems - Chapter 6**Problem 1:** Prove by contrapositive: If n^2 is divisible by 3, then n is divisible by 3.**Solution:**

Contrapositive: If n is not divisible by 3, then n^2 is not divisible by 3.

If n not divisible by 3, then $n = 3k + 1$ or $n = 3k + 2$.

Case 1: $n = 3k + 1 \Rightarrow n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ (not divisible by 3)

Case 2: $n = 3k + 2 \Rightarrow n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ (not divisible by 3)

Problem 2: Prove by contradiction: There is no largest integer.**Solution:**

Assume there is a largest integer, call it N .

Consider $N + 1$. This is an integer and $N + 1 > N$.

This contradicts that N is the largest.

Therefore, no largest integer exists.

Chapter 7: Mathematical Induction

Mathematical induction is a powerful proof technique for proving statements about all positive integers. It is particularly useful for proving properties of sequences, summations, and recursively defined objects.

7.1 Principle of Mathematical Induction

Definition 7.1: Principle of Mathematical Induction

To prove that a statement $P(n)$ is true for all positive integers n :

1. **Base Case:** Prove $P(1)$ is true
2. **Inductive Step:** Assume $P(k)$ is true for some $k \geq 1$ (inductive hypothesis), then prove $P(k + 1)$ is true

If both steps are satisfied, $P(n)$ is true for all $n \geq 1$.

Example 7.1: Induction Proof

Theorem: $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ for all $n \geq 1$

Proof by Induction:

Base Case ($n = 1$):

$$\text{LHS} = \sum_{i=1}^1 i = 1$$

$$\text{RHS} = \frac{1(1+1)}{2} = 1 \quad \checkmark$$

Inductive Step:

Assume $\sum_{i=1}^k i = \frac{k(k+1)}{2}$ (Inductive Hypothesis)

Prove: $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\text{LHS} = \sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \text{RHS} \quad \checkmark$$

Therefore, by induction, the formula holds for all $n \geq 1$. ■

★ **Key Point Trick: Induction Proof Structure**

The "Domino Effect" Analogy:

- Base case = First domino falls
- Inductive step = If domino k falls, domino $k + 1$ falls
- Conclusion = All dominos fall

Template for Induction:

1. State: "We prove by mathematical induction"
2. Base case: Verify $P(1)$ explicitly
3. Inductive hypothesis: "Assume $P(k)$ is true..."
4. Show $P(k + 1)$ using $P(k)$
5. Conclude: "By induction, $P(n)$ holds for all $n \geq 1$ "

7.2 Strong Induction

Definition 7.2: Strong Induction

To prove $P(n)$ for all $n \geq 1$ using **strong induction**:

1. **Base Case:** Prove $P(1)$ is true
2. **Inductive Step:** Assume $P(1), P(2), \dots, P(k)$ are all true, then prove $P(k + 1)$ is true

The inductive hypothesis is stronger — we assume all previous cases, not just $P(k)$.

Example 7.2: Strong Induction

Theorem: Every integer $n \geq 2$ can be written as a product of primes.

Proof by Strong Induction:

Base Case ($n = 2$): 2 is prime, so it's a product of one prime. ✓

Inductive Step:

Assume all integers from 2 to k can be written as products of primes.

Consider $k + 1$:

- If $k + 1$ is prime, done.
- If $k + 1$ is composite, then $k + 1 = ab$ where $2 \leq a, b \leq k$.
- By inductive hypothesis, a and b are products of primes.
- Therefore, $k + 1 = ab$ is a product of primes. ✓

By strong induction, every integer $n \geq 2$ is a product of primes. ■

★ **Key Point Trick: When to Use Strong Induction**

Use strong induction when:

- The proof of $P(k + 1)$ depends on multiple previous cases
- Dealing with recursive definitions where F_n depends on multiple previous terms
- Proving properties about factorization, sequences with multiple-term recurrences

Regular vs Strong: Strong induction is more powerful but regular induction is simpler when it works.

7.3 Structural Induction

Definition 7.3: Structural Induction

Structural induction is used to prove properties about recursively defined sets (like trees, formulas, or well-formed expressions). It proves the property for:

1. **Base case:** All minimal (atomic) elements
2. **Inductive step:** If the property holds for component elements, it holds for elements constructed from them

Example 7.3: Structural Induction

Theorem: Every well-formed formula of propositional logic has an equal number of left and right parentheses.

Proof by Structural Induction:

Base Case: Atomic propositions (like p, q) have 0 parentheses. ✓

Inductive Step:

Assume formulas ϕ and ψ have equal left and right parentheses.

- $\neg\phi$: Same parentheses as ϕ . Balanced. ✓
- $(\phi \wedge \psi)$: Adds one '(' and one ')'. Still balanced. ✓
- Similarly for $\vee, \rightarrow, \leftrightarrow$

By structural induction, all well-formed formulas have balanced parentheses. ■

Common Mistake: Induction Errors

Missing base case: Always verify $P(1)$ explicitly!

Circular reasoning: Don't assume $P(k + 1)$ to prove $P(k + 1)$

Wrong inductive hypothesis: Must assume $P(k)$, not $P(k + 1)$ or $P(n)$

Incomplete step: Must explicitly show how $P(k)$ leads to $P(k + 1)$

Chapter 7 Summary

- **Mathematical Induction:** Base case + Inductive step proves $P(n)$ for all n
- **Base Case:** Prove $P(1)$ explicitly
- **Inductive Hypothesis:** Assume $P(k)$ is true
- **Inductive Step:** Prove $P(k + 1)$ using $P(k)$
- **Strong Induction:** Assume $P(1), \dots, P(k)$ to prove $P(k + 1)$
- **Structural Induction:** For recursively defined structures

Practice Problems - Chapter 7

Problem 1: Prove by induction: $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Base ($n = 1$): LHS = 1, RHS = $\frac{1 \cdot 2 \cdot 3}{6} = 1 \checkmark$

Inductive Step: Assume true for $n = k$. Prove for $n = k + 1$.

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \checkmark \end{aligned}$$

Chapter 8: Counting Principles

Counting principles are fundamental tools in combinatorics that help us determine the number of elements in sets without explicit enumeration. This chapter covers two of the most important counting principles.

8.1 Inclusion-Exclusion Principle

Definition 8.1: Inclusion-Exclusion Principle

For two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

General Pattern: Add singles, subtract pairs, add triples, subtract quadruples, ...

Example 8.1: Inclusion-Exclusion

In a class of 50 students:

- 30 study Mathematics ($|M| = 30$)
- 25 study Physics ($|P| = 25$)
- 15 study both ($|M \cap P| = 15$)

How many study Math or Physics?

$$|M \cup P| = |M| + |P| - |M \cap P| = 30 + 25 - 15 = 40$$

Students studying neither: $50 - 40 = 10$

★ **Key Point Trick: Inclusion-Exclusion Memory Aid**

"Add all, subtract overlaps, add back triple overlaps, ..."

The principle corrects for overcounting:

- Elements in $A \cap B$ counted twice \rightarrow subtract once
- Elements in $A \cap B \cap C$ counted 3 times, subtracted 3 times \rightarrow add back once

Formula pattern: $\sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \dots$

8.2 Pigeonhole Principle

Definition 8.2: Pigeonhole Principle (Basic)

If n items are put into m containers, with $n > m$, then at least one container must contain more than one item.

Theorem 8.1: Generalized Pigeonhole Principle

If n items are put into m containers, then at least one container contains at least $\lceil n/m \rceil$ items.

Example 8.2: Pigeonhole Principle

Problem: In any group of 13 people, at least 2 share a birth month.

Solution: 13 people (pigeons), 12 months (pigeonholes). Since $13 > 12$, at least 2 people share a birth month.

Problem: How many people needed to guarantee at least 3 share a birth month?

Solution: We need $\lceil n/12 \rceil \geq 3$, so $n/12 > 2$, thus $n > 24$. Need at least 25 people.

Example 8.3: Pigeonhole Application

Theorem: Among any 5 points inside a unit square, some pair is at distance at most $\sqrt{2}/2$.

Proof:

Divide the unit square into 4 smaller squares of side $1/2$.

5 points (pigeons), 4 small squares (pigeonholes).

By pigeonhole principle, at least 2 points are in the same small square.

Maximum distance in a small square = diagonal = $\sqrt{(1/2)^2 + (1/2)^2} = \sqrt{2}/2$. ■

★ **Key Point Trick: Applying Pigeonhole Principle****Step-by-step:**

- 1 Identify what represents "pigeons" (items to distribute)
- 2 Identify what represents "pigeonholes" (categories)
- 3 Verify $n > m$ (more pigeons than holes)
- 4 Apply the principle to conclude

Chapter 8 Summary

- **Inclusion-Exclusion:** $|A \cup B| = |A| + |B| - |A \cap B|$
- **Corrects overcounting:** Subtract intersections, add back triple intersections
- **Pigeonhole Principle:** n items in m containers, $n > m \rightarrow$ some container has ≥ 2 items
- **Generalized:** At least $\lceil n/m \rceil$ items in some container

Practice Problems - Chapter 8

Problem 1: Among 100 people, at least how many were born in the same month?

Solution:

By generalized pigeonhole: $\lceil 100/12 \rceil = \lceil 8.33\dots \rceil = 9$

At least 9 people share a birth month.

Problem 2: How many integers from 1 to 100 are divisible by 3 or 5?

Solution:

Let A = divisible by 3, B = divisible by 5

$$|A| = \lfloor 100/3 \rfloor = 33, |B| = \lfloor 100/5 \rfloor = 20$$

$$|A \cap B| = \lfloor 100/15 \rfloor = 6 \text{ (divisible by both = divisible by 15)}$$

$$|A \cup B| = 33 + 20 - 6 = 47$$

Chapter 9: Permutations and Combinations

Permutations and combinations are fundamental counting techniques that answer the question "how many ways?" in different scenarios. Understanding when to use each is crucial for solving counting problems.

9.1 Permutations

Definition 9.1: Permutation

A **permutation** is an ordered arrangement of objects. The number of permutations of n distinct objects taken r at a time is:

$$P(n, r) = \frac{n!}{(n - r)!}$$

Theorem 9.1: Permutations of n Objects

The number of ways to arrange all n distinct objects is:

$$P(n, n) = n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

Example 9.1: Permutations

Problem: How many ways to arrange 5 people in a line?

Solution: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Problem: How many ways to select and arrange 3 people from 8?

Solution: $P(8, 3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336$

9.2 Combinations

Definition 9.2: Combination

A **combination** is an unordered selection of objects. The number of combinations of n distinct objects taken r at a time is:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Theorem 9.2: Combination Properties

1. $\binom{n}{r} = \binom{n}{n-r}$ (Symmetry)
2. $\binom{n}{0} = \binom{n}{n} = 1$
3. $\binom{n}{1} = \binom{n}{n-1} = n$

Example 9.2: Combinations

Problem: How many ways to choose a committee of 3 from 10 people?

Solution: $\binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$

Problem: How many ways to choose 5 cards from a 52-card deck?

Solution: $\binom{52}{5} = \frac{52!}{5!47!} = 2,598,960$

★ **Key Point Trick: Permutation vs Combination**

	Permutation	Combination
Order matters?	YES	NO
Formula	$P(n, r) = \frac{n!}{(n-r)!}$	$C(n, r) = \frac{n!}{r!(n-r)!}$
When to use	Arrangements, sequences	Selections, groups
Key word	"Arrange"	"Choose" or "Select"

Relationship: $P(n, r) = C(n, r) \times r!$ (choose first, then arrange)

9.3 Permutations with Repetition

Definition 9.3: Permutations with Repeated Elements

The number of distinct permutations of n objects where there are n_1 identical objects of type 1, n_2 identical objects of type 2, ..., n_k identical objects of type k is:

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

where $n_1 + n_2 + \dots + n_k = n$.

Example 9.3: Permutations with Repetition

Problem: How many distinct arrangements of "MISSISSIPPI"?

Letters: M(1), I(4), S(4), P(2) — total 11 letters

Solution: $\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = \frac{39,916,800}{1 \cdot 24 \cdot 24 \cdot 2} = 34,650$

9.4 Combinations with Repetition

Definition 9.4: Combinations with Repetition

The number of ways to choose r objects from n types of objects, where repetition is allowed and order doesn't matter, is:

$$C(n + r - 1, r) = \binom{n + r - 1}{r}$$

Example 9.4: Combinations with Repetition

Problem: How many ways to choose 5 donuts from 8 types?

Solution: $\binom{8+5-1}{5} = \binom{12}{5} = \frac{12!}{5! \cdot 7!} = 792$

★ Key Point Trick: Stars and Bars Method

For combinations with repetition, use the **stars and bars** visualization:

- Stars (*) represent the r objects being chosen
- Bars (|) separate the n types (need $n - 1$ bars)

Example: Choosing 5 donuts from 3 types (A, B, C)

|*| means 2 of type A, 3 of type B, 0 of type C

Total positions = 5 stars + 2 bars = 7 positions

Choose 2 positions for bars (or 5 for stars): $\binom{7}{2} = \binom{7}{5} = 21$

Chapter 9 Summary

- **Permutation** $P(n, r)$: Ordered selection, $\frac{n!}{(n-r)!}$
- **Combination** $C(n, r)$: Unordered selection, $\frac{n!}{r!(n-r)!}$
- **Permutation with repetition**: $\frac{n!}{n_1! \cdot n_2! \cdot \dots}$
- **Combination with repetition**: $\binom{n+r-1}{r}$
- **Key distinction**: Does order matter?

Practice Problems - Chapter 9

Problem 1: How many 4-letter passwords from 26 letters (no repetition)?

Solution:

Order matters (password "ABCD" \neq "DCBA")

$$P(26, 4) = 26 \times 25 \times 24 \times 23 = 358,800$$

Problem 2: How many ways to choose 6 questions from 10 on an exam?

Solution:

Order doesn't matter

$$\binom{10}{6} = \binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

Chapter 10: The Binomial Theorem

The Binomial Theorem provides a formula for expanding powers of binomials and reveals deep connections between combinatorics and algebra. It is one of the most important results in discrete mathematics.

10.1 Binomial Coefficients

Definition 10.1: Binomial Coefficient

The **binomial coefficient** $\binom{n}{r}$ (read "n choose r") counts the number of ways to choose r elements from a set of n elements:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Theorem 10.1: Properties of Binomial Coefficients

1. $\binom{n}{0} = \binom{n}{n} = 1$
2. $\binom{n}{r} = \binom{n}{n-r}$ (Symmetry)
3. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ (Pascal's Identity)

10.2 The Binomial Theorem

Theorem 10.2: The Binomial Theorem

For any non-negative integer n :

$$\begin{aligned} (a+b)^n &= \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r \\ &= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n} b^n \end{aligned}$$

Example 10.1: Binomial ExpansionExpand $(x + y)^4$:

$$\begin{aligned}(x + y)^4 &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

Example 10.2: Finding Specific Terms**Problem:** Find the coefficient of x^5y^3 in $(x + y)^8$ **Solution:** The term is $\binom{8}{3}x^5y^3$ (or $\binom{8}{5}x^5y^3$)

$$\text{Coefficient} = \binom{8}{3} = \frac{8!}{3!5!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

★ Key Point Trick: Binomial Theorem Patterns**Pattern for $(a + b)^n$:**

- There are $n + 1$ terms
- Powers of a decrease: $n, n - 1, n - 2, \dots, 0$
- Powers of b increase: $0, 1, 2, \dots, n$
- Sum of exponents in each term equals n
- Coefficients are $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$

Quick formula for middle term: For $(a + b)^n$, the $(r + 1)$ th term is $\binom{n}{r}a^{n-r}b^r$

10.3 Pascal's Triangle

Definition 10.2: Pascal's Triangle

Pascal's Triangle is a triangular array where each entry is the sum of the two entries above it. Row n contains the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$.

Example 10.3: Pascal's Triangle

```

Row 0:      1
Row 1:     1  1
Row 2:    1  2  1
Row 3:   1  3  3  1
Row 4:  1  4  6  4  1
Row 5:  1  5 10 10  5  1

```

Each number = sum of two numbers above it (Pascal's Identity)

Theorem 10.3: Sum of Binomial Coefficients

$$\sum_{r=0}^n \binom{n}{r} = 2^n$$

This counts all subsets of an n -element set.

★ Key Point Trick: Binomial Identities

Identity	Formula
Sum of coefficients	$\sum_{r=0}^n \binom{n}{r} = 2^n$
Alternating sum	$\sum_{r=0}^n (-1)^r \binom{n}{r} = 0$
Sum of even terms	$\binom{n}{0} + \binom{n}{2} + \dots = 2^{n-1}$
Sum of odd terms	$\binom{n}{1} + \binom{n}{3} + \dots = 2^{n-1}$

Proof technique: Set $a = b = 1$ or $a = 1, b = -1$ in binomial theorem

Chapter 10 Summary

- **Binomial Theorem:** $(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$
- **Binomial Coefficient:** $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- **Pascal's Identity:** $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
- **Pascal's Triangle:** Visual representation of binomial coefficients
- **Sum identity:** $\sum_{r=0}^n \binom{n}{r} = 2^n$

Practice Problems - Chapter 10**Problem 1:** Expand $(2x - 3)^4$ **Solution:**

$$\begin{aligned}
 (2x - 3)^4 &= \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-3) + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)(-3)^3 \\
 &= 16x^4 + 4 \cdot 8x^3 \cdot (-3) + 6 \cdot 4x^2 \cdot 9 + 4 \cdot 2x \cdot (-27) + 81 \\
 &= 16x^4 - 96x^3 + 216x^2 - 216x + 81
 \end{aligned}$$

Problem 2: Find the coefficient of x^3 in $(1 + 2x)^7$ **Solution:**

$$\text{Term with } x^3: \binom{7}{3}(1)^4(2x)^3 = \binom{7}{3} \cdot 8x^3$$

$$\binom{7}{3} = \frac{7!}{3!4!} = 35$$

$$\text{Coefficient} = 35 \times 8 = 280$$

Chapter 11: Discrete Probability

Probability theory provides a mathematical framework for analyzing random phenomena. Discrete probability deals with countable sample spaces and is fundamental to algorithm analysis, cryptography, machine learning, and many other areas of computer science.

11.1 Basic Probability Concepts

Definition 11.1: Sample Space and Events

- **Sample Space (S):** The set of all possible outcomes of an experiment
- **Event (E):** A subset of the sample space
- **Outcome:** An element of the sample space

Definition 11.2: Probability

For a finite sample space S with equally likely outcomes, the **probability** of event E is:

$$P(E) = \frac{|E|}{|S|} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Example 11.1: Basic Probability

Experiment: Roll a fair six-sided die

Sample space: $S = \{1, 2, 3, 4, 5, 6\}$, $|S| = 6$

Event E: Roll an even number = $\{2, 4, 6\}$, $|E| = 3$

$$P(E) = \frac{3}{6} = \frac{1}{2}$$

Theorem 11.1: Probability Properties

1. $0 \leq P(E) \leq 1$ for any event E
2. $P(S) = 1$ (some outcome must occur)
3. $P(\emptyset) = 0$
4. $P(\overline{E}) = 1 - P(E)$ (complement rule)
5. $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

11.2 Conditional Probability**Definition 11.3: Conditional Probability**

The **conditional probability** of event E given that event F has occurred is:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

where $P(F) > 0$.

Example 11.2: Conditional Probability

Draw a card from a standard deck.

E = card is a king, F = card is a heart

$$P(E) = \frac{4}{52} = \frac{1}{13}, P(F) = \frac{13}{52} = \frac{1}{4}$$

$$P(E \cap F) = P(\text{king of hearts}) = \frac{1}{52}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/52}{13/52} = \frac{1}{13}$$

Given it's a heart, probability it's a king = $\frac{1}{13}$ (only one king of hearts among 13 hearts)

Definition 11.4: Independence

Events E and F are **independent** if:

$$P(E \cap F) = P(E) \cdot P(F)$$

Equivalently: $P(E|F) = P(E)$ (knowing F doesn't change probability of E)

★ **Key Point Trick: Independence Check**

To check if E and F are independent:

1. Calculate $P(E)$, $P(F)$, and $P(E \cap F)$
2. Check if $P(E \cap F) = P(E) \cdot P(F)$
3. If equal → Independent; If not → Dependent

Common independent events: Coin flips, die rolls (different trials)

Common dependent events: Drawing without replacement

11.3 Bayes' Theorem

Theorem 11.2: Bayes' Theorem

For events E and F with $P(E) > 0$ and $P(F) > 0$:

$$P(F|E) = \frac{P(E|F) \cdot P(F)}{P(E)}$$

Extended form (for partition F_1, F_2, \dots, F_n):

$$P(F_i|E) = \frac{P(E|F_i) \cdot P(F_i)}{\sum_{j=1}^n P(E|F_j) \cdot P(F_j)}$$

Example 11.3: Bayes' Theorem Application**Medical Testing Problem:**

- Disease prevalence: $P(D) = 0.01$ (1% have disease)
- Test sensitivity: $P(T|D) = 0.95$ (95% true positive)
- Test specificity: $P(\neg T|\neg D) = 0.90$ (90% true negative)

Question: If you test positive, what's $P(D|T)$?

Solution:

$$P(T|\neg D) = 1 - 0.90 = 0.10$$

$$P(T) = P(T|D)P(D) + P(T|\neg D)P(\neg D)$$

$$= 0.95 \times 0.01 + 0.10 \times 0.99 = 0.0095 + 0.099 = 0.1085$$

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{0.95 \times 0.01}{0.1085} \approx 0.0876$$

Only about 8.8% chance of having the disease despite positive test!

★ **Key Point Trick: Bayes' Theorem Memory Aid**

"Flip-Flop Formula"

Bayes' theorem lets you "flip" conditional probabilities:

- Know $P(E|F) \rightarrow$ Find $P(F|E)$

Formula pattern: $\frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$

Tree diagram method: Draw probability tree, then use:

$$P(F|E) = \frac{\text{path through F to E}}{\text{all paths to E}}$$

11.4 Random Variables

Definition 11.5: Random Variable

A **random variable** is a function that assigns a numerical value to each outcome in a sample space.

Definition 11.6: Expected Value

The **expected value** (or mean) of a random variable X is:

$$E(X) = \sum_x x \cdot P(X = x)$$

Definition 11.7: Variance

The **variance** of a random variable X is:

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = E[(X - E(X))^2]$$

The **standard deviation** is $\sigma = \sqrt{\text{Var}(X)}$.

Example 11.4: Expected Value

Roll a fair die. Let X = value showing.

$$\begin{aligned} E(X) &= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \\ &= \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5 \end{aligned}$$

★ **Key Point Trick: Linearity of Expectation**

Theorem: For any random variables X and Y :

$$E(X + Y) = E(X) + E(Y)$$

This holds **even if X and Y are dependent!**

Application: Expected number of heads in 10 coin flips = $10 \times 0.5 = 5$

Chapter 11 Summary

- **Probability:** $P(E) = \frac{|E|}{|S|}$ for equally likely outcomes
- **Conditional:** $P(E|F) = \frac{P(E \cap F)}{P(F)}$
- **Independence:** $P(E \cap F) = P(E) \cdot P(F)$
- **Bayes' Theorem:** $P(F|E) = \frac{P(E|F)P(F)}{P(E)}$
- **Expected Value:** $E(X) = \sum x \cdot P(X = x)$
- **Linearity:** $E(X + Y) = E(X) + E(Y)$ always

Practice Problems - Chapter 11

Problem 1: Two cards drawn without replacement. P(both aces)?

Solution:

$$P(\text{first ace}) = \frac{4}{52} = \frac{1}{13}$$

$$P(\text{second ace} | \text{first ace}) = \frac{3}{51} = \frac{1}{17}$$

$$P(\text{both aces}) = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221}$$

Problem 2: Expected value of sum when rolling two dice?

Solution:

$$\text{By linearity: } E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

Chapter 12: Recurrence Relations

Recurrence relations provide a way to define sequences recursively. They are essential in analyzing algorithms, counting problems, and modeling discrete processes. This chapter covers methods for solving linear recurrence relations.

12.1 Introduction to Recurrence Relations

Definition 12.1: Recurrence Relation

A **recurrence relation** for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous terms of the sequence.

Definition 12.2: Linear Recurrence Relation

A **linear recurrence relation of degree k with constant coefficients** has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$$

where c_1, c_2, \dots, c_k are constants and $c_k \neq 0$.

- If $f(n) = 0$: **homogeneous**
- If $f(n) \neq 0$: **non-homogeneous**

Example 12.1: Famous Recurrence Relations

Sequence	Recurrence	Initial Conditions
Fibonacci	$F_n = F_{n-1} + F_{n-2}$	$F_0 = 0, F_1 = 1$
Factorial	$n! = n \cdot (n - 1)!$	$0! = 1$
Powers of 2	$a_n = 2a_{n-1}$	$a_0 = 1$

12.2 Solving Linear Recurrence Relations

Theorem 12.1: Solving Homogeneous Linear Recurrences

For $a_n = c_1 a_{n-1} + c_2 a_{n-2}$:

1. Form the **characteristic equation**: $r^2 = c_1 r + c_2$ or $r^2 - c_1 r - c_2 = 0$
2. Find roots r_1, r_2
3. **Case 1 (Distinct roots)**: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
4. **Case 2 (Repeated root)**: $a_n = (\alpha_1 + \alpha_2 n) r^n$
5. Use initial conditions to find α_1, α_2

Example 12.2: Solving Fibonacci Recurrence

Solve: $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1$

Solution:

Characteristic equation: $r^2 = r + 1$ or $r^2 - r - 1 = 0$

Roots: $r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

Let $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$ (golden ratio) and $\psi = \frac{1-\sqrt{5}}{2} \approx -0.618$

General solution: $F_n = \alpha_1 \phi^n + \alpha_2 \psi^n$

Using initial conditions:

- $F_0 = 0: \alpha_1 + \alpha_2 = 0$
- $F_1 = 1: \alpha_1 \phi + \alpha_2 \psi = 1$

Solving: $\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = -\frac{1}{\sqrt{5}}$

Binet's Formula: $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$

★ Key Point Trick: Solving Recurrence Relations

Step-by-step method:

- 1 Write characteristic equation by replacing a_n with r^n
- 2 Solve for roots
- 3 Write general solution based on root type
- 4 Use initial conditions to find constants

Root types:

- Distinct real roots: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- Repeated root: $a_n = (\alpha_1 + \alpha_2 n) r^n$
- Complex roots: Use Euler's formula or keep in trig form

12.3 Divide-and-Conquer Relations

Definition 12.3: Divide-and-Conquer Recurrence

A divide-and-conquer recurrence has the form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$, $b > 1$, and $f(n)$ is the cost of dividing and combining.

Theorem 12.2: Master Theorem

For $T(n) = aT(n/b) + O(n^d)$ where $a \geq 1$, $b > 1$:

1. If $a > b^d$: $T(n) = O(n^{\log_b a})$
2. If $a = b^d$: $T(n) = O(n^d \log n)$
3. If $a < b^d$: $T(n) = O(n^d)$

Example 12.3: Master Theorem Applications

Recurrence	a	b	d	Case	Solution
Merge Sort: $T(n) = 2T(n/2) + O(n)$	2	2	1	$a = b^d$	$O(n \log n)$
Binary Search: $T(n) = T(n/2) + O(1)$	1	2	0	$a = b^d$	$O(\log n)$
Strassen: $T(n) = 7T(n/2) + O(n^2)$	7	2	2	$a > b^d$	$O(n^{\log_2 7}) \approx O(n^{2.81})$

★ Key Point Trick: Master Theorem Quick Reference

Compare a vs b^d :

- **More subproblems** ($a > b^d$): Work dominated by leaves $\rightarrow O(n^{\log_b a})$
- **Balanced** ($a = b^d$): Work at all levels equal $\rightarrow O(n^d \log n)$
- **More combine work** ($a < b^d$): Work dominated by root $\rightarrow O(n^d)$

Memory: "Compare a to b-to-the-d"

Chapter 12 Summary

- **Recurrence relation:** Defines sequence terms using previous terms
- **Characteristic equation:** Replace a_n with r^n to find roots
- **General solution:** Based on root type (distinct, repeated, complex)
- **Initial conditions:** Determine constants in general solution
- **Master Theorem:** Solves divide-and-conquer recurrences
- **Compare a vs b^d :** Determines which case applies

Practice Problems - Chapter 12**Problem 1:** Solve $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_0 = 1, a_1 = 4$ **Solution:**

Characteristic: $r^2 - 5r + 6 = 0$

$(r - 2)(r - 3) = 0$, so $r = 2, 3$

General: $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 3^n$

$a_0 = 1: \alpha_1 + \alpha_2 = 1$

$a_1 = 4: 2\alpha_1 + 3\alpha_2 = 4$

Solving: $\alpha_1 = -1, \alpha_2 = 2$

$a_n = -2^n + 2 \cdot 3^n = 2 \cdot 3^n - 2^n$

Problem 2: Use Master Theorem for $T(n) = 4T(n/2) + n^2$ **Solution:**

$a = 4, b = 2, d = 2$

$b^d = 2^2 = 4 = a$

Case 2: $T(n) = O(n^2 \log n)$

Final Revision Sheet

Critical Formulas and Concepts

Logic

- $p \rightarrow q \equiv \neg p \vee q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (De Morgan's)
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's)
- $\neg \forall x P(x) \equiv \exists x \neg P(x)$
- $\neg \exists x P(x) \equiv \forall x \neg P(x)$

Sets

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|\mathcal{P}(A)| = 2^{|A|}$
- $|A \times B| = |A| \cdot |B|$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Functions

- Injective: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
- Surjective: Range = Codomain
- Bijective: Both injective and surjective (has inverse)
- $(g \circ f)(x) = g(f(x))$

Summation

- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=0}^n r^i = \frac{r^{n+1}-1}{r-1} \quad (r \neq 1)$

Counting

- $P(n, r) = \frac{n!}{(n-r)!}$ (permutations)
- $C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ (combinations)
- With repetition: $\binom{n+r-1}{r}$
- Pigeonhole: n items in m containers, $n > m \rightarrow$ some container has ≥ 2

Binomial Theorem

- $(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$
- $\sum_{r=0}^n \binom{n}{r} = 2^n$
- $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

Probability

- $P(E) = \frac{|E|}{|S|}$
- $P(E|F) = \frac{P(E \cap F)}{P(F)}$
- Bayes': $P(F|E) = \frac{P(E|F)P(F)}{P(E)}$
- $E(X) = \sum x \cdot P(X = x)$
- $E(X + Y) = E(X) + E(Y)$

Recurrence Relations

- Characteristic equation: $r^2 = c_1 r + c_2$
- Distinct roots: $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
- Repeated root: $a_n = (\alpha_1 + \alpha_2 n) r^n$
- Master Theorem: Compare a vs b^d

★ Quick Decision Guide

When you see...	Use...
"Arrange" or "Order matters"	Permutation $P(n, r)$
"Choose" or "Select" or "Committee"	Combination $C(n, r)$
"With repetition allowed"	Stars and bars
"At least one"	Complement: $1 - P(\text{none})$
"Given that"	Conditional probability
"Prove for all n"	Mathematical induction
"If p then q"	Direct proof or contrapositive

Additional Practice Problems

Comprehensive Practice Set

Problem 1 (Logic): Construct a truth table for $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$

Solution:

This is the hypothetical syllogism, which is a tautology (always true).

Problem 2 (Sets): Prove: $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Solution:

Both sides represent elements in exactly one of A or B (symmetric difference).

Problem 3 (Functions): Prove $f(x) = x^3$ is bijective on \mathbb{R} .

Solution:

Injective: $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ (cube root is unique)

Surjective: For any $y \in \mathbb{R}$, $x = \sqrt[3]{y}$ satisfies $f(x) = y$

Problem 4 (Counting): How many ways to arrange 8 people in a circle?

Solution:

Circular permutations: $(n - 1)! = 7! = 5040$

(Fix one person, arrange the rest)

Problem 5 (Probability): A family has 2 children. Given that at least one is a boy, what's P(both boys)?

Solution:

Sample space given condition: $\{BB, BG, GB\}$ (not GG)

$$P(\text{both boys}) = \frac{1}{3}$$

Problem 6 (Induction): Prove $7^n - 2^n$ is divisible by 5 for all $n \geq 1$.

Solution:

Base: $7^1 - 2^1 = 5$, divisible by 5. ✓

Assume $7^k - 2^k = 5m$ for some integer m .

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k = 7(5m + 2^k) - 2 \cdot 2^k \\ &= 35m + 7 \cdot 2^k - 2 \cdot 2^k = 35m + 5 \cdot 2^k = 5(7m + 2^k) \end{aligned}$$

Divisible by 5. ✓

Problem 7 (Recurrence): Solve $a_n = 4a_{n-1} - 4a_{n-2}$ with $a_0 = 1, a_1 = 4$.

Solution:

Characteristic: $r^2 - 4r + 4 = 0$

$(r - 2)^2 = 0$, so $r = 2$ (repeated)

General: $a_n = (\alpha_1 + \alpha_2 n) \cdot 2^n$

$a_0 = 1$: $\alpha_1 = 1$

$a_1 = 4$: $(1 + \alpha_2) \cdot 2 = 4$, so $\alpha_2 = 1$

$a_n = (1 + n) \cdot 2^n$