Let's get started with...

Logic!

Logic

- Crucial for mathematical reasoning
- Important for program design
- Used for designing electronic circuitry

- (Propositional)Logic is a system based on propositions.
- A proposition is a (declarative) statement that is either true or false (not both).
- We say that the truth value of a proposition is either true (T) or false (F).
- Corresponds to 1 and 0 in digital circuits

The Statement/Proposition Game

"Elephants are bigger than mice."

Is this a statement?	yes
Is this a proposition?	yes
What is the truth value of the proposition?	true

The Statement/Proposition Game

“520 < 111”

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

The Statement/Proposition Game

"$y > 5$"

Is this a statement? yes

Is this a proposition? no

Its truth value depends on the value of y, but this value is not specified.

We call this type of statement a propositional function or open sentence.

The Statement/Proposition Game

"Today is January 27 and $99 < 5$."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? false

The Statement/Proposition Game

"Please do not fall asleep."

Is this a statement? no

It's a request.

Is this a proposition? no

Only statements can be propositions.

The Statement/Proposition Game

"If the moon is made of cheese, then I will be rich."

Is this a statement? yes

Is this a proposition? yes

What is the truth value of the proposition? probably true

The Statement/Proposition Game

"$x < y$ if and only if $y > x$."

Is this a statement? yes

Is this a proposition? yes

... because its truth value does not depend on specific values of x and y.

What is the truth value of the proposition? true

Combining Propositions

As we have seen in the previous examples, one or more propositions can be combined to form a single compound proposition.

We formalize this by denoting propositions with letters such as p, q, r, s, and introducing several logical operators or logical connectives.

Logical Operators (Connectives)

We will examine the following logical operators:

- Negation (NOT, $\neg$)
- Conjunction (AND, $\wedge$)
- Disjunction (OR, $\vee$)
- Exclusive-or (XOR, $\oplus$)
- Implication (if – then, $\rightarrow$)
- Biconditional (if and only if, $\leftrightarrow$)

Truth tables can be used to show how these operators can combine propositions to compound propositions.

Negation (NOT)

Unary Operator, Symbol: $\neg$

P	$\neg P$
true (T)	false (F)
false (F)	true (T)

Conjunction (AND)

Binary Operator, Symbol: $\wedge$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction (OR)

Binary Operator, Symbol: $\vee$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive Or (XOR)

Binary Operator, Symbol: $\oplus$

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (if - then)

Binary Operator, Symbol: $\rightarrow$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional (if and only if)

Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Statements and Operators

Statements and operators can be combined in any way to form new statements.

P	Q	$\neg P$	$\neg Q$	$(\neg P)\vee(\neg Q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Statements and Operations

Statements and operators can be combined in any way to form new statements.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
- When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
- School is closed if more than 2 feet of snow falls or if the wind chill is below -100.

Exercises

- To take discrete mathematics, you must have taken calculus or a course in computer science.
  - P: take discrete mathematics
  - Q: take calculus
  - R: take a course in computer science
- $P \rightarrow Q \vee R$
- Problem with proposition R
  - What if I want to represent "take CMSC201"?

Exercises

- When you buy a new car from Acme Motor Company, you get $2000 back in cash or a 2% car loan.
  - P: buy a car from Acme Motor Company
  - Q: get $2000 cash back
  - R: get a 2% car loan
- $P \rightarrow Q \oplus R$
- Why use XOR here? – example of ambiguity of natural languages

Exercises

- School is closed if more than 2 feet of snow falls or if the wind chill is below -100.
  - P: School is closed
  - Q: 2 feet of snow falls
  - R: wind chill is below -100
- $Q \wedge R \rightarrow P$
- Precedence among operators:

  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

Equivalent Statements

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent, since they have the same truth table, or put it in another way, $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$ is always true.

Tautologies and Contradictions

A tautology is a statement that is always true.

Examples:

- $R \vee (\neg R)$
- $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

A contradiction is a statement that is always false.

Examples:

- $R \wedge (\neg R)$
- $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q))$

The negation of any tautology is a contradiction, and the negation of any contradiction is a tautology.

Equivalence

Definition: two propositional statements S1 and S2 are said to be (logically) equivalent, denoted S1 $\equiv$ S2 if

- They have the same truth table, or
- S1 $\Leftrightarrow$ S2 is a tautology

Equivalence can be established by

- Constructing truth tables
- Using equivalence laws (Table 5 in Section 1.2)

Equivalence

Equivalence laws

- Identity laws, $P \wedge T \equiv P$,
- Domination laws, $P \wedge F \equiv F$,
- Idempotent laws, $P \wedge P \equiv P$,
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P$,
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$,
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$,
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$
- Law with implication $P \rightarrow Q \equiv \neg P \vee Q$

Exercises

- Show that $P \to Q \equiv \neg P \vee Q$: by truth table
- Show that $(P \to Q) \wedge (P \to R) \equiv P \to (Q \wedge R)$: by equivalence laws (q20, p27):
  - Law with implication on both sides
  - Distribution law on LHS

Summary, Sections 1.1, 1.2

- Proposition
  - Statement, Truth value,
  - Proposition, Propositional symbol, Open proposition
- Operators
  - Define by truth tables
  - Composite propositions
  - Tautology and contradiction
- Equivalence of propositional statements
  - Definition
  - Proving equivalence (by truth table or equivalence laws)

Propositional Functions & Predicates

Propositional function (open sentence):

statement involving one or more variables,

e.g.: $x-3 > 5$.

Let us call this propositional function $P(x)$, where P is the predicate and x is the variable.

What is the truth value of $P(2)$? false

What is the truth value of $P(8)$? false

What is the truth value of $P(9)$? true

When a variable is given a value, it is said to be instantiated

Truth value depends on value of variable

Propositional Functions

Let us consider the propositional function $Q(x, y, z)$ defined as:

$x + y = z$.

Here, Q is the predicate and x, y, and z are the variables.

What is the truth value of $Q(2, 3, 5)$? true

What is the truth value of $Q(0, 1, 2)$? false

What is the truth value of $Q(9, -9, 0)$? true

A propositional function (predicate) becomes a proposition when all its variables are instantiated.

Propositional Functions

Other examples of propositional functions

Person(x), which is true if x is a person

- Person(Socrates) = T
- Person(dolly-the-sheep) = F

CSCourse(x), which is true if x is a computer science course

- CSCourse(CMSC201) = T
- CSCourse(MATH155) = F

How do we say

- All humans are mortal
- One CS course

Universal Quantification

Let P(x) be a predicate (propositional function).

Universally quantified sentence:

For all x in the universe of discourse P(x) is true.

Using the universal quantifier $\forall$:

$\forall x\, P(x)$ "for all x P(x)" or "for every x P(x)"

(Note: $\forall x\, P(x)$ is either true or false, so it is a proposition, not a propositional function.)

Universal Quantification

Example: Let the universe of discourse be all people

$S(x)$: x is a UMBC student.

$G(x)$: x is a genius.

What does $\forall x\ (S(x) \rightarrow G(x))$ mean ?

"If x is a UMBC student, then x is a genius." or

"All UMBC students are geniuses."

If the universe of discourse is all UMBC students, then the same statement can be written as

$\forall x\ G(x)$

Existential Quantification

Existentially quantified sentence:

There exists an x in the universe of discourse for which $P(x)$ is true.

Using the existential quantifier $\exists$:

$\exists x\, P(x)$ "There is an x such that $P(x)$."

"There is at least one x such that $P(x)$."

(Note: $\exists x\, P(x)$ is either true or false, so it is a proposition, but no propositional function.)

Existential Quantification

Example:

P(x): x is a UMBC professor.

G(x): x is a genius.

What does $\exists x\ (P(x) \land G(x))$ mean ?

"There is an x such that x is a UMBC professor and x is a genius."

or

"At least one UMBC professor is a genius."

Quantification

Another example:

Let the universe of discourse be the real numbers.

What does $\forall x \exists y \, (x + y = 320)$ mean ?

"For every x there exists a y so that x + y = 320."

Is it true? yes

Is it true for the natural numbers? no

Disproof by Counterexample

A counterexample to $\forall x\ P(x)$ is an object c so that P(c) is false.

Statements such as $\forall x\ (P(x) \rightarrow Q(x))$ can be disproved by simply providing a counterexample.

Statement: "All birds can fly."
Disproved by counterexample: Penguin.

Negation

$\neg(\forall x\ P(x))$ is logically equivalent to $\exists x\ (\neg P(x))$.

$\neg(\exists x\ P(x))$ is logically equivalent to $\forall x\ (\neg P(x))$.

See Table 2 in Section 1.3.

This is de Morgan's law for quantifiers

Negation

Examples

Not all roses are red

$\neg\forall x\ (Rose(x) \rightarrow Red(x))$

$\exists x\ (Rose(x) \land \neg Red(x))$

Nobody is perfect

$\neg\exists x\ (Person(x) \land Perfect(x))$

$\forall x\ (Person(x) \rightarrow \neg Perfect(x))$

Nested Quantifier

- A predicate can have more than one variables.
  - $S(x, y, z)$: z is the sum of x and y
  - $F(x, y)$: x and y are friends
- We can quantify individual variables in different ways
  - $\forall x, y, z\ (S(x, y, z) \rightarrow (x <= z \wedge y <= z))$
  - $\exists x\ \forall y\ \forall z\ (F(x, y) \wedge F(x, z) \wedge (y != z) \rightarrow \neg F(y, z)$

Nested Quantifier

Exercise: translate the following English sentence into logical expression

"There is a rational number in between every pair of distinct rational numbers"

Use predicate $Q(x)$, which is true when x is a rational number

$$\forall x,y\, (Q(x) \wedge Q(y) \wedge (x < y) \rightarrow \exists u\, (Q(u) \wedge (x < u) \wedge (u < y)))$$

Summary, Sections 1.3, 1.4

- Propositional functions (predicates)
- Universal and existential quantifiers, and the duality of the two
- When predicates become propositions
  - All of its variables are instantiated
  - All of its variables are quantified
- Nested quantifiers
  - Quantifiers with negation
- Logical expressions formed by predicates, operators, and quantifiers

Let's proceed to…

Mathematical Reasoning

Mathematical Reasoning

We need **mathematical reasoning** to

- determine whether a mathematical argument is correct or incorrect and
- construct mathematical arguments.

Mathematical reasoning is not only important for conducting **proofs** and **program verification**, but also for **artificial intelligence** systems (drawing logical inferences from knowledge and facts).

We focus on **deductive** proofs

Terminology

An **axiom** is a basic assumption about mathematical structure that needs no proof.

- Things known to be true (facts or proven theorems)
- Things believed to be true but cannot be proved

We can use a **proof** to demonstrate that a particular statement is true. A proof consists of a sequence of statements that form an argument.

The steps that connect the statements in such a sequence are the **rules of inference**.

Cases of incorrect reasoning are called **fallacies**.

Terminology

A **theorem** is a statement that can be shown to be true.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **corollary** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

Proofs

A theorem often has two parts
- Conditions (premises, hypotheses)
- conclusion

A correct (deductive) proof is to establish that
- If the conditions are true then the conclusion is true
- I.e., Conditions $\rightarrow$ conclusion is a tautology

Often there are missing pieces between conditions and conclusion. Fill it by an argument
- Using conditions and axioms
- Statements in the argument connected by proper rules of inference

Rules of Inference

Rules of inference provide the justification of the steps used in a proof.

One important rule is called modus ponens or the law of detachment. It is based on the tautology $(p \wedge (p \rightarrow q)) \rightarrow q$. We write it in the following way:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

The two hypotheses p and $p \rightarrow q$ are written in a column, and the conclusion below a bar, where $\therefore$ means "therefore".

Rules of Inference

The general form of a rule of inference is:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

The rule states that if p_1 **and** p_2 **and** … **and** p_n are all true, then q is true as well.

Each rule is an established tautology of

$$p_1 \land p_2 \land \ldots \land p_n \rightarrow q$$

These rules of inference can be used in any mathematical argument and do not require any proof.

Rules of Inference

$$\frac{p}{\therefore p \vee q}$$ Addition

$$\frac{p \wedge q}{\therefore p}$$ Simplification

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$$ Conjunction

$$\frac{\begin{array}{c} \neg q \\ p \to q \end{array}}{\therefore \neg p}$$ Modus tollens

$$\frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{\therefore p \to r}$$ Hypothetical syllogism (chaining)

$$\frac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{\therefore q}$$ Disjunctive syllogism (resolution)

Arguments

Just like a rule of inference, an **argument** consists of one or more hypotheses (or premises) and a conclusion.

We say that an argument is **valid**, if whenever all its hypotheses are true, its conclusion is also true.

However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.

Proof: show that hypotheses $\rightarrow$ conclusion is true using rules of inference

Arguments

Example:

"If 101 is divisible by 3, then 101^2 is divisible by 9. 101 is divisible by 3. Consequently, 101^2 is divisible by 9."

Although the argument is **valid**, its conclusion is **incorrect**, because one of the hypotheses is false ("101 is divisible by 3.").

If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Arguments

Which rule of inference was used in the last argument?

p: "101 is divisible by 3."

q: "101^2 is divisible by 9."

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Modus ponens

Unfortunately, one of the hypotheses (p) is false. Therefore, the conclusion q is incorrect.

Arguments

Another example:

"If it rains today, then we will not have a barbeque today. If we do not have a barbeque today, then we will have a barbeque tomorrow. Therefore, if it rains today, then we will have a barbeque tomorrow."

This is a **valid** argument: If its hypotheses are true, then its conclusion is also true.

Arguments

Let us formalize the previous argument:

p: "It is raining today."

q: "We will not have a barbecue today."

r: "We will have a barbecue tomorrow."

So the argument is of the following form:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Hypothetical syllogism

Arguments

Another example:

Gary is either intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: "Gary is intelligent."
a: "Gary is a good actor."
c: "Gary can count from 1 to 10."

Arguments

i: "Gary is intelligent."
a: "Gary is a good actor."
c: "Gary can count from 1 to 10."

Step 1:	$\neg c$	Hypothesis
Step 2:	$i \rightarrow c$	Hypothesis
Step 3:	$\neg i$	Modus tollens Steps 1 & 2
Step 4:	$a \vee i$	Hypothesis
Step 5:	a	Disjunctive Syllogism Steps 3 & 4

Conclusion: a ("Gary is a good actor.")

Arguments

Yet another example:

If you listen to me, you will pass CS 320.
You passed CS 320.
Therefore, you have listened to me.

Is this argument valid?

No, it assumes $((p \to q) \wedge q) \to p$.
This statement is not a tautology. It is false if p is false and q is true.

Rules of Inference for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$

Universal instantiation

$$\frac{P(c) \text{ for an arbitrary } c \in U}{\therefore \forall x\, P(x)}$$

Universal generalization

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c \in U}$$

Existential instantiation

$$\frac{P(c) \text{ for some element } c \in U}{\therefore \exists x\, P(x)}$$

Existential generalization

Rules of Inference for Quantified Statements

Example:

Every UMB student is a genius.
George is a UMB student.
Therefore, George is a genius.

U(x): "x is a UMB student."
G(x): "x is a genius."

Rules of Inference for Quantified Statements

The following steps are used in the argument:

Step 1:	$\forall x\, (U(x) \rightarrow G(x))$	Hypothesis
Step 2:	$U(George) \rightarrow G(George)$	Univ. instantiation using Step 1
Step 3:	$U(George)$	Hypothesis
Step 4:	$G(George)$	Modus ponens using Steps 2 & 3

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$ Universal instantiation

Proving Theorems

Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: *Give a direct proof of the theorem "If n is odd, then n^2 is odd."*

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd).

Proving Theorems

n is odd.

Then $n = 2k + 1$, where k is an integer.

Consequently, $n^2 = (2k + 1)^2$.

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

Since n^2 can be written in this form, it is odd.

Proving Theorems

Indirect proof:

An implication $p \rightarrow q$ is equivalent to its **contra-positive** $\neg q \rightarrow \neg p$. Therefore, we can prove $p \rightarrow q$ by showing that whenever q is false, then p is also false.

Example: Give an indirect proof of the theorem "If 3n + 2 is odd, then n is odd."

Idea: Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false (3n + 2 is even).

Proving Theorems

n is even.

Then $n = 2k$, where k is an integer.

It follows that
$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Therefore, $3n + 2$ is even.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If $3n + 2$ is odd, then n is odd).

Proving Theorems

Indirect Proof is a special case of proof by contradiction

Suppose n is even (negation of the conclusion).

Then $n = 2k$, where k is an integer.

It follows that

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Therefore, $3n + 2$ is even.

However, this is a contradiction since $3n + 2$ is given to be odd, so the conclusion (n is odd) holds.

Another Example on Proof

Anyone performs well is either intelligent or a good actor.
If someone is intelligent, then he/she can count from 1 to 10.
Gary performs well.
Gary can only count from 1 to 3.
Therefore, not everyone is both intelligent and a good actor

- P(x): x performs well
- I(x): x is intelligent
- A(x): x is a good actor
- C(x): x can count from 1 to 10

Another Example on Proof

Hypotheses:

1. Anyone performs well is either intelligent or a good actor.
   $\forall x\, (P(x) \rightarrow I(x) \vee A(x))$
2. If someone is intelligent, then he/she can count from 1 to 10.
   $\forall x\, (I(x) \rightarrow C(x))$
3. Gary performs well.
   $P(G)$
4. Gary can only count from 1 to 3.
   $\neg C(G)$

Conclusion: not everyone is both intelligent and a good actor
$\neg \forall x (I(x) \wedge A(x))$

Another Example on Proof

Direct proof:

Step 1: $\forall x\, (P(x) \rightarrow I(x) \vee A(x))$	Hypothesis
Step 2: $P(G) \rightarrow I(G) \vee A(G)$	Univ. Inst. Step 1
Step 3: $P(G)$	Hypothesis
Step 4: $I(G) \vee A(G)$	Modus ponens Steps 2 & 3
Step 5: $\forall x\, (I(x) \rightarrow C(x))$	Hypothesis
Step 6: $I(G) \rightarrow C(G)$	Univ. inst. Step5
Step 7: $\neg C(G)$	Hypothesis
Step 8: $\neg I(G)$	Modus tollens Steps 6 & 7
Step 9: $\neg I(G) \vee \neg A(G)$	Addition Step 8
Step 10: $\neg(I(G) \wedge A(G))$	Equivalence Step 9
Step 11: $\exists x \neg(I(x) \wedge A(x))$	Exist. general. Step 10
Step 12: $\neg \forall x\, (I(x) \wedge A(x))$	Equivalence Step 11

Conclusion: $\neg \forall x\, (I(x) \wedge A(x))$, not everyone is both intelligent and a good actor.

Summary, Section 1.5

- Terminology (axiom, theorem, conjecture, argument, etc.)
- Rules of inference (Tables 1 and 2)
- Valid argument (hypotheses and conclusion)
- Construction of valid argument using rules of inference
  - For each rule used, write down and the statements involved in the proof
- Direct and indirect proofs
  - Other proof methods (e.g., induction, pigeon hole) will be introduced in later chapters