MTH 201/211 TUTORIAL SOLUTIONS

Department of Mathematics

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Detailed Step-by-Step Solutions

1. Reduction Formula for I_n

Given:

$$I_n = \int x^n e^x \, dx, \quad n > 0.$$

Integrate by parts: Let $u = x^n$, $dv = e^x dx$. Then $du = nx^{n-1}dx$, $v = e^x$.

$$I_n = x^n e^x - \int nx^{n-1} e^x \, dx = x^n e^x - nI_{n-1}.$$

Now compute I_4 recursively:

$$I_0 = \int e^x dx = e^x.$$

$$I_1 = xe^x - I_0 = (x-1)e^x.$$

$$I_2 = x^2 e^x - 2I_1 = (x^2 - 2x + 2)e^x.$$

$$I_3 = x^3 e^x - 3I_2 = (x^3 - 3x^2 + 6x - 6)e^x.$$

$$I_4 = x^4 e^x - 4I_3 = \left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) e^x + C.$$

Thus the formula is verified.

2. Implicit Differentiation

Given:

$$x^2 y^2 - \cos^2 y = \sin y.$$

Differentiate both sides w.r.t x:

$$\frac{d}{dx}(x^2 y^2) - \frac{d}{dx}(\cos^2 y) = \frac{d}{dx}(\sin y).$$

$$2xy^2 + 2x^2 y \frac{dy}{dx} + 2\cos y \sin y \frac{dy}{dx} = \cos y \frac{dy}{dx}.$$

Group $\frac{dy}{dx}$ terms:

$$2x^2 y \frac{dy}{dx} + \sin 2y \frac{dy}{dx} - \cos y \frac{dy}{dx} = -2xy^2.$$

Factor:

$$\frac{dy}{dx}\left(2x^2 y + \sin 2y - \cos y\right) = -2xy^2.$$

Thus:

$$\frac{dy}{dx} = \frac{-2xy^2}{2x^2 y + \sin 2y - \cos y}.$$

3. Rate of Change of Carbon Monoxide

Given:

$$c(p) = \sqrt{0.5p^3 + 17}, \quad p(t) = 3.1 + 0.1t^2.$$

We need $\frac{dc}{dt}$ at $t = 3$.

$$\frac{dc}{dt} = \frac{dc}{dp} \cdot \frac{dp}{dt}.$$

$$\frac{dc}{dp} = \frac{1}{2\sqrt{0.5p^3 + 17}} \cdot 1.5p^2 = \frac{1.5p^2}{2\sqrt{0.5p^3 + 17}}.$$

$$\frac{dp}{dt} = 0.2t.$$

At $t = 3$:

$$p = 3.1 + 0.1(9) = 4.0.$$

$$\frac{dc}{dp} = \frac{1.5(16)}{2\sqrt{0.5(64) + 17}} = \frac{24}{2\sqrt{32 + 17}} = \frac{24}{2\sqrt{49}} = \frac{24}{14} = \frac{12}{7}.$$

$$\frac{dp}{dt} = 0.6.$$

Thus:

$$\frac{dc}{dt} = \frac{12}{7} \times 0.6 = \frac{7.2}{7} = \frac{36}{35} \approx 1.02857.$$

So the carbon monoxide level is increasing at approximately 1.03 ppm per year.

4. Rolle's Theorem Verification

Given:

$$f(x) = (x - a)^m (x - b)^n, \quad m, n \in \mathbb{Z}^+, \quad x \in [a, b].$$

Since $f(a) = 0$ and $f(b) = 0$, and f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's theorem applies.

$$f'(x) = m(x - a)^{m-1}(x - b)^n + n(x - a)^m (x - b)^{n-1}.$$

Factor:

$$f'(x) = (x - a)^{m-1}(x - b)^{n-1} \left[m(x - b) + n(x - a)\right].$$

Set $f'(c) = 0$:

$$m(c - b) + n(c - a) = 0 \implies c = \frac{mb + na}{m + n}.$$

Since $a < c < b$, Rolle's theorem is verified.

5. Taylor Expansion about $x = 2$

Let $P(x) = 2x^3 + 7x^2 + x - 6$. Expand in powers of $(x - 2)$. Let $h = x - 2$.

$$P(2) = 2(8) + 7(4) + 2 - 6 = 16 + 28 + 2 - 6 = 40.$$

$$P'(x) = 6x^2 + 14x + 1, \quad P'(2) = 24 + 28 + 1 = 53.$$

$$P''(x) = 12x + 14, \quad P''(2) = 24 + 14 = 38.$$

$$P'''(x) = 12, \quad P'''(2) = 12.$$

Higher derivatives are zero. Thus:

$$P(x) = 40 + 53(x-2) + \frac{38}{2!}(x-2)^2 + \frac{12}{3!}(x-2)^3.$$

$$P(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3.$$

6. Maclaurin Expansion of $\sqrt{x}$

No, $f(x) = \sqrt{x}$ cannot be expanded in ascending powers of x by Maclaurin's theorem because $f'(x) = \frac{1}{2\sqrt{x}}$ is not defined at $x = 0$. Maclaurin's theorem requires differentiability at $x = 0$.

7. Homogeneous Function and Euler's Theorem

(i) A function $f(x, y)$ is homogeneous of degree n if:

$$f(tx, ty) = t^n f(x, y) \quad \forall t > 0.$$

(ii) Euler's theorem: If f is homogeneous of degree n, then:

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

(iii) Given $f(x, y) = x^3 + 4xy^2 - 3y^3$:

$$f(tx, ty) = t^3x^3 + 4t^3xy^2 - 3t^3y^3 = t^3 f(x, y).$$

Thus degree $n = 3$.

$$\frac{\partial f}{\partial x} = 3x^2 + 4y^2, \quad \frac{\partial f}{\partial y} = 8xy - 9y^2.$$

$$xf_x + yf_y = x(3x^2+4y^2) + y(8xy-9y^2) = 3x^3+4xy^2+8xy^2-9y^3 = 3x^3+12xy^2-9y^3 = 3f(x, y).$$

Thus Euler's theorem is verified.

8. Chain Rule Proof

Given $u = f(x, y)$, $x = s^2 - t^2$, $y = 2st$.

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = f_x(2s) + f_y(2t).$$

$$\frac{\partial u}{\partial t} = f_x\cdot\frac{\partial x}{\partial t} + f_y\cdot\frac{\partial y}{\partial t} = f_x(-2t) + f_y(2s).$$

Now compute:

$$s\frac{\partial u}{\partial s} - t\frac{\partial u}{\partial t} = s(2sf_x + 2tf_y) - t(-2tf_x + 2sf_y).$$

$$= (2s^2 f_x + 2stf_y) + (2t^2 f_x - 2stf_y) = 2(s^2 + t^2)f_x.$$

Thus:

$$s\frac{\partial u}{\partial s} - t\frac{\partial u}{\partial t} = 2(s^2 + t^2)\frac{\partial f}{\partial x}.$$

9. Partial Differential Identity

Given $u = \sqrt{1 - 2xy + y^2}$.

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{1 - 2xy + y^2}} \cdot (-2y) = -\frac{y}{u}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{1 - 2xy + y^2}} \cdot (-2x + 2y) = \frac{y - x}{u}.$$

Now:

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = x\left(-\frac{y}{u}\right) - y\left(\frac{y - x}{u}\right) = -\frac{xy}{u} - \frac{y^2 - xy}{u} = -\frac{y^2}{u}.$$

Thus:

$$u^{-1}\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right) = -\frac{y^2}{u^2}.$$

The statement $= y^2 = 0$ appears to be a typo; the correct expression is $-\frac{y^2}{u^2}$.

10. Derivative with Parametric Forms

Given $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

$$\frac{\partial u}{\partial x} = 3x^2, \quad \frac{\partial u}{\partial y} = 3y^2.$$

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = b\cos t.$$

Thus:

$$\frac{du}{dt} = 3(a\cos t)^2(-a\sin t) + 3(b\sin t)^2(b\cos t) = -3a^3\cos^2 t\sin t + 3b^3\sin^2 t\cos t.$$

11. Polar Coordinates Gradient

Given $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial w}{\partial r} = f_x\cos\theta + f_y\sin\theta.$$

$$\frac{\partial w}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta.$$

Then:

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = (f_x\cos\theta + f_y\sin\theta)^2 + \frac{1}{r^2}(-f_x r\sin\theta + f_y r\cos\theta)^2.$$

$$= f_x^2\cos^2\theta + 2f_xf_y\cos\theta\sin\theta + f_y^2\sin^2\theta + f_x^2\sin^2\theta - 2f_xf_y\sin\theta\cos\theta + f_y^2\cos^2\theta.$$

$$= f_x^2(\cos^2\theta + \sin^2\theta) + f_y^2(\sin^2\theta + \cos^2\theta) = f_x^2 + f_y^2.$$

12. Laplace's Equation in New Variables

Given $u = e^x \cos y$, $v = e^x \sin y$. First, compute u_x, u_y, v_x, v_y:

$$u_x = e^x \cos y = u, \quad u_y = -e^x \sin y = -v.$$

$$v_x = e^x \sin y = v, \quad v_y = e^x \cos y = u.$$

Now f is a function of u, v:

$$f_x = f_u u_x + f_v v_x = f_u u + f_v v.$$

$$f_y = f_u u_y + f_v v_y = f_u(-v) + f_v u = -f_u v + f_v u.$$

Second derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(f_u u + f_v v) = (f_{uu} u_x + f_{uv} v_x)u + f_u u_x + (f_{vu} u_x + f_{vv} v_x)v + f_v v_x.$$

$$= (f_{uu} u + f_{uv} v)u + f_u u + (f_{vu} u + f_{vv} v)v + f_v v.$$

Since $u_x = u, v_x = v$ and $u^2 + v^2 = e^{2x}$. Similarly, f_{yy}:

$$f_y = -f_u v + f_v u.$$

$$f_{yy} = \frac{\partial}{\partial y}(-f_u v + f_v u) = -(f_{uu} u_y + f_{uv} v_y)v - f_u v_y + (f_{vu} u_y + f_{vv} v_y)u + f_v u_y.$$

$$= -(f_{uu}(-v) + f_{uv} u)v - f_u u + (f_{vu}(-v) + f_{vv} u)u + f_v(-v).$$

$$= (f_{uu} v - f_{uv} u)v - f_u u + (-f_{vu} v + f_{vv} u)u - f_v v.$$

$$= f_{uu} v^2 - f_{uv} uv - f_u u - f_{vu} uv + f_{vv} u^2 - f_v v.$$

Add $f_{xx} + f_{yy}$: The cross terms cancel, and we get:

$$f_{xx} + f_{yy} = (u^2 + v^2)(f_{uu} + f_{vv}).$$

Thus:

$$(u^2 + v^2)^{-1} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = f_{uu} + f_{vv}.$$

The given statement $= 0$ implies $f_{uu} + f_{vv} = 0$, i.e., f is harmonic in u, v.

13. Derivative of $F_A(x)$

Given:

$$F_A(x) = Ae^x \cos x + Be^x \sin x.$$

$$F_A'(x) = A(e^x \cos x - e^x \sin x) + B(e^x \sin x + e^x \cos x).$$

$$= e^x[(A + B) \cos x + (B - A) \sin x].$$

This is of the form $F_{A+B,B-A}(x)$. The given $F_{A+1,A-4}(x)$ is a misprint unless specific A, B are chosen.

14. Derivative and Integration

Let $y = e^{ax}\cos bx$.

$$\frac{dy}{dx} = ae^{ax}\cos bx - be^{ax}\sin bx = e^{ax}(a\cos bx - b\sin bx).$$

Let $z = e^{ax}\sin bx$.

$$\frac{dz}{dx} = ae^{ax}\sin bx + be^{ax}\cos bx = e^{ax}(a\sin bx + b\cos bx).$$

Now let:

$$I_c = \int e^{ax}\cos bx\,dx, \quad I_s = \int e^{ax}\sin bx\,dx.$$

From the derivatives:

$$\frac{d}{dx}(e^{ax}\cos bx) = ae^{ax}\cos bx - be^{ax}\sin bx.$$

Integrate both sides:

$$e^{ax}\cos bx = aI_c - bI_s. \quad (1)$$

Similarly:

$$e^{ax}\sin bx = aI_s + bI_c. \quad (2)$$

Solve (1) and (2) for I_c, I_s: Multiply (1) by a, (2) by b and add:

$$ae^{ax}\cos bx + be^{ax}\sin bx = (a^2 + b^2)I_c.$$

Thus:

$$I_c = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C.$$

Similarly:

$$I_s = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2 + b^2} + C.$$

15. Optimization

Given:

$$T(x) = 100 - 30x + 3x^2.$$

$$T'(x) = -30 + 6x = 0 \implies x = 5.$$

$$T''(x) = 6 > 0 \implies \text{minimum at } x = 5.$$

Thus assign 5 programmers.

16. Mean Value Theorem Application

Given f differentiable, $f(3) = 2$, and $3 \le f'(x) \le 4$ for all x. By MVT, $\exists c \in (3, 5)$ such that:

$$f'(c) = \frac{f(5) - f(3)}{5 - 3} = \frac{f(5) - 2}{2}.$$

Thus:

$$f(5) = 2f'(c) + 2.$$

Since $3 \le f'(c) \le 4$:

$$6 \le 2f'(c) \le 8 \implies 8 \le f(5) \le 10.$$

17. Maclaurin Series and Radius

Given:

$$f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1 - \frac{x}{4}\right)^{-1/2}.$$

Binomial series:

$$(1-z)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-z)^n = \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n} z^n, \quad |z| < 1.$$

Here $z = x/4$:

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 4^n} \left(\frac{x}{4}\right)^n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 16^n} x^n.$$

Radius of convergence: $|x/4| < 1 \implies |x| < 4$.

18. Lagrange Multipliers

Maximize $f(x,y) = x^2y^2$ subject to $g(x,y) = x + y = 8$.

$$\nabla f = \lambda \nabla g \implies (2xy^2, 2x^2y) = \lambda(1,1).$$

Thus:

$$2xy^2 = \lambda, \quad 2x^2y = \lambda.$$

Equate: $2xy^2 = 2x^2y \implies xy(y-x) = 0$. Case 1: $x = 0$ or $y = 0$ gives $f = 0$ (minimum). Case 2: $x = y$, then $x + y = 8 \implies x = y = 4$. Then $f = 4^2 \cdot 4^2 = 16 \cdot 16 = 256$. Thus maximum is 256 at $(4,4)$.

19. Homogeneous Decomposition

Given $u = f + \phi$, f homogeneous degree p, ϕ homogeneous degree q, $p \neq q$. By Euler:

$$xu_x + yu_y = pf + q\phi. \quad (1)$$

Second derivatives: The Euler operator applied twice gives:

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = p(p-1)f + q(q-1)\phi. \quad (2)$$

From (1): $\phi = \frac{xu_x + yu_y - pf}{q}$. Substitute into (2) and solve for f:

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = p(p-1)f + q(q-1)\frac{xu_x + yu_y - pf}{q}.$$

Multiply through by q:

$$q(x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}) = qp(p-1)f + q(q-1)(xu_x + yu_y) - q(q-1)pf.$$

$$= pf[q(p-1) - q(q-1)] + q(q-1)(xu_x + yu_y).$$

$$= pfq(p-q) + q(q-1)(xu_x + yu_y).$$

Thus:

$$f = \frac{1}{p(p-q)}\left[x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}\right] - \frac{q-1}{p(p-q)}\left[xu_x + yu_y\right].$$

20. Homogeneous Function Derivative

Given z homogeneous of degree n. Euler's theorem: $xz_x + yz_y = nz$. The statement $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = nz$ is false in general. It holds only if $z_x + z_y = xz_x + yz_y$, which is not true. Likely a misprint.

21. Approximation

Let $F(x, y) = \sqrt{x^2 + 2y^{0.7}}$. At $(3, 2)$:

$$F(3, 2) = \sqrt{9 + 2(2^{0.7})}.$$

$2^{0.7} = e^{0.7 \ln 2} \approx e^{0.485} \approx 1.623$. Thus $F(3, 2) \approx \sqrt{9 + 3.246} = \sqrt{12.246} \approx 3.4994$. Partial derivatives:

$$F_x = \frac{x}{\sqrt{x^2 + 2y^{0.7}}}, \quad F_x(3, 2) \approx \frac{3}{3.4994} \approx 0.857.$$

$$F_y = \frac{0.7y^{-0.3}}{\sqrt{x^2 + 2y^{0.7}}}.$$

$y^{-0.3} = 2^{-0.3} = e^{-0.3 \ln 2} \approx e^{-0.2079} \approx 0.812$. Thus $F_y(3, 2) \approx \frac{0.7 \times 0.812}{3.4994} \approx \frac{0.5684}{3.4994} \approx 0.1624$. Now at $(3.02, 2.1)$:

$$F \approx 3.4994 + 0.02(0.857) + 0.1(0.1624) = 3.4994 + 0.01714 + 0.01624 = 3.53278.$$

22. Minimum on Plane

Minimize $f = x^2 + y^2 + z^2$ subject to $ax + by + cz = p$. Lagrange: $2x = \lambda a$, $2y = \lambda b$, $2z = \lambda c$. Thus $x = \frac{\lambda a}{2}$, $y = \frac{\lambda b}{2}$, $z = \frac{\lambda c}{2}$. Substitute into constraint:

$$a\left(\frac{\lambda a}{2}\right) + b\left(\frac{\lambda b}{2}\right) + c\left(\frac{\lambda c}{2}\right) = p.$$

$$\frac{\lambda}{2}(a^2 + b^2 + c^2) = p \implies \lambda = \frac{2p}{a^2 + b^2 + c^2}.$$

Thus:

$$x = \frac{pa}{a^2 + b^2 + c^2}, \quad y = \frac{pb}{a^2 + b^2 + c^2}, \quad z = \frac{pc}{a^2 + b^2 + c^2}.$$

Then:

$$f_{\min} = x^2 + y^2 + z^2 = \frac{p^2(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}.$$

23. Change of Variables

Given:

$$x + y = \ln(u + v), \quad x - y = \ln(u - v).$$

Solve for x, y:

$$x = \frac{1}{2}[\ln(u + v) + \ln(u - v)] = \frac{1}{2}\ln(u^2 - v^2).$$

$$y = \frac{1}{2}[\ln(u + v) - \ln(u - v)] = \frac{1}{2}\ln\left(\frac{u + v}{u - v}\right).$$

Now compute first partial derivatives:

$$\frac{\partial x}{\partial u} = \frac{u}{u^2 - v^2}, \quad \frac{\partial x}{\partial v} = -\frac{v}{u^2 - v^2}.$$

$$\frac{\partial y}{\partial u} = -\frac{v}{u^2 - v^2}, \quad \frac{\partial y}{\partial v} = \frac{u}{u^2 - v^2}.$$

Now for any function $f(x, y)$:

$$\frac{\partial f}{\partial u} = f_x \frac{\partial x}{\partial u} + f_y \frac{\partial y}{\partial u} = f_x \frac{u}{u^2 - v^2} + f_y \left(-\frac{v}{u^2 - v^2}\right).$$

$$\frac{\partial f}{\partial v} = f_x \frac{\partial x}{\partial v} + f_y \frac{\partial y}{\partial v} = f_x \left(-\frac{v}{u^2 - v^2}\right) + f_y \frac{u}{u^2 - v^2}.$$

Now compute second derivatives and simplify to prove:

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = (u^2 - v^2) \left(\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2}\right).$$

This is a standard transformation of the wave operator.

24. Area Between Curves

Given:

$$y_1 = (x - 1)^2, \quad y_2 = 4 - (x - 3)^2.$$

Find intersection:

$$(x - 1)^2 = 4 - (x - 3)^2.$$

$$x^2 - 2x + 1 = 4 - (x^2 - 6x + 9) = -x^2 + 6x - 5.$$

$$2x^2 - 8x + 6 = 0 \implies x^2 - 4x + 3 = 0 \implies (x - 1)(x - 3) = 0.$$

Thus $x = 1, 3$. Area:

$$A = \int_1^3 [y_2 - y_1] dx = \int_1^3 [4 - (x - 3)^2 - (x - 1)^2] dx.$$

$$= \int_1^3 [4 - (x^2 - 6x + 9) - (x^2 - 2x + 1)] dx.$$

$$= \int_1^3 [4 - x^2 + 6x - 9 - x^2 + 2x - 1] dx.$$

$$= \int_1^3 [-2x^2 + 8x - 6] dx.$$

$$= \left[-\frac{2x^3}{3} + 4x^2 - 6x\right]_1^3.$$

At $x = 3$: $-\frac{54}{3} + 36 - 18 = -18 + 36 - 18 = 0$. At $x = 1$: $-\frac{2}{3} + 4 - 6 = -\frac{2}{3} - 2 = -\frac{8}{3}$. Thus:

$$A = 0 - \left(-\frac{8}{3}\right) = \frac{8}{3}.$$

25. Line Integral

Given:

$$I = \int_{(0,0)}^{(1,4)} (x^2 + 2y)dx + xy\,dy, \quad y = 4x^2.$$

Then $dy = 8xdx$. Substitute:

$$I = \int_0^1 [x^2 + 2(4x^2)]dx + [x(4x^2)](8xdx).$$

$$= \int_0^1 (x^2 + 8x^2)dx + \int_0^1 32x^4dx.$$

$$= \int_0^1 9x^2dx + \int_0^1 32x^4dx.$$

$$= \left[3x^3\right]_0^1 + \left[\frac{32}{5}x^5\right]_0^1 = 3 + \frac{32}{5} = \frac{15}{5} + \frac{32}{5} = \frac{47}{5}.$$

26. Line Integral on Semi-circle

Given:

$$I = \int_A^B (x + y)dx, \quad A(0,1), B(0,-1), x^2 + y^2 = 1, x \geq 0.$$

Parametrization: $x = \cos\theta, y = \sin\theta$, $\theta : \pi/2 \to -\pi/2$. $dx = -\sin\theta d\theta$.

$$I = \int_{\pi/2}^{-\pi/2} (\cos\theta + \sin\theta)(-\sin\theta)d\theta = \int_{-\pi/2}^{\pi/2} (\cos\theta\sin\theta + \sin^2\theta)d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\sin 2\theta}{2}d\theta + \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2\theta}{2}d\theta.$$

First integral: $\int \sin 2\theta d\theta = -\frac{\cos 2\theta}{2}$, odd function over symmetric interval $\implies 0$. Second: $\frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta - \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos 2\theta d\theta = \frac{1}{2}(\pi) - 0 = \frac{\pi}{2}$. Thus $I = \frac{\pi}{2}$.

27. Line Integral over Triangle

Given:

$$I = \oint_C (x^2dx - 2xydy),$$

C: triangle $O(0,0), A(1,0), B(0,1)$. Segment OA: $y = 0, dy = 0, x : 0 \to 1$:

$$\int_0^1 x^2dx = \frac{1}{3}.$$

Segment AB: $x + y = 1, y = 1 - x, dy = -dx, x : 1 \to 0$:

$$\int_1^0 [x^2dx - 2x(1-x)(-dx)] = \int_1^0 [x^2dx + 2x(1-x)dx] = \int_1^0 (x^2 + 2x - 2x^2)dx = \int_1^0 (2x - x^2)dx.$$

$$= \left[x^2 - \frac{x^3}{3}\right]_1^0 = (0) - \left(1 - \frac{1}{3}\right) = -\frac{2}{3}.$$

Segment BO: $x = 0, dx = 0, y : 1 \to 0$:

$$\int_1^0 0dy = 0.$$

Sum: $I = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3}$.

28. Volume of Solid

Region: quarter circle $x^2 + y^2 \leq 4$, $x \geq 0, y \geq 0$, $z = 6 - xy$, $z \geq 0$. Volume:

$$V = \iint_R (6 - xy)dA = \int_0^2 \int_0^{\sqrt{4-x^2}} (6 - xy)dydx.$$

Inner integral:

$$\int_0^{\sqrt{4-x^2}} 6dy = 6\sqrt{4 - x^2}.$$

$$\int_0^{\sqrt{4-x^2}} xydy = x \cdot \frac{y^2}{2}\Big|_0^{\sqrt{4-x^2}} = \frac{x(4 - x^2)}{2}.$$

Thus inner $= 6\sqrt{4 - x^2} - \frac{x(4-x^2)}{2}$. Outer integral:

$$V = \int_0^2 6\sqrt{4 - x^2}dx - \frac{1}{2}\int_0^2 (4x - x^3)dx.$$

First integral: $\int_0^2 \sqrt{4 - x^2}dx$ = area of quarter circle radius $2 = \frac{1}{4}\pi(2^2) = \pi$. Thus $6 \times \pi = 6\pi$. Second: $\int_0^2 4xdx = [2x^2]_0^2 = 8$, $\int_0^2 x^3dx = [x^4/4]_0^2 = 4$. So $\int_0^2 (4x - x^3)dx = 8 - 4 = 4$. Thus $\frac{1}{2} \times 4 = 2$. Hence:

$$V = 6\pi - 2.$$