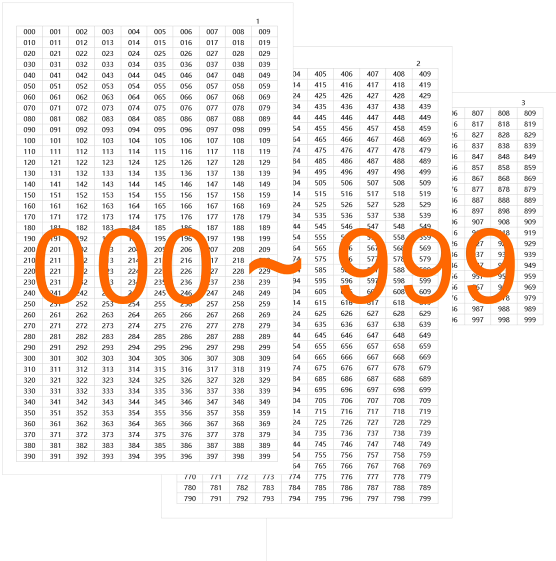
PERMUTATIONS AND COMBINATIONS

With and Without Repetition

permutations & combinations

3-digit PIN example (000–999)



Friends on a bench

Ice cream cones with 5 flavors

Quick challenge!

How many ways can I arrange these 3 cards?

Introduction

In how many different ways can we arrange a group of objects?

How many possible selections can we make from a larger set?

Does the order of selection matter?

Are we allowed to repeat elements?

Introduction (contd.)

Permutations - arrangements where order matters

Combinations - selections where order does not matter

Motivation and relevance

- **Probability and Statistics**
- **Computer Science and Data Science**
- **Biology and Medicine**
- **Economics and Social Sciences**

Learning objectives

- Explain the difference between permutations and combinations
- Determine whether order matters in a given problem
- Compute permutations **without repetition**
- Compute permutations **with repetition**
- Compute combinations **without repetition**
- Compute combinations **with repetition**
- Apply the appropriate formulas to real-world problems

Fundamental counting principle (rule of product)

If one event can occur in **m** ways and a second independent event can occur in **n** ways, then the two events together can occur in

$$m \times n \text{ ways}.$$

Example

Suppose you have **3 shirts**, and **2 pairs of trousers**. The number of different outfits you can form is:

$$3 \times 2 = 6$$

This simple idea will now be extended to more complex counting problems.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Permutations (order matters)

A permutation is an arrangement of objects where the order is important.

For example, given a set {A, B, C}, the permutations ABC, ACB, BAC, BCA, CAB, and CBA are all distinct because they arrange the elements in different orders.

Permutations without repetition

Problem Setup

Suppose we have **n distinct objects**, and we want to arrange **r** of them **without reusing any object**.

Derivation

- The first position can be filled in **n** ways
- The second position in **(n – 1)** ways
- The third position in **(n – 2)** ways
- Continue until **r** positions are filled

Formula

The number of permutations of **n objects taken r at a time** is:

$$P(n,r) = \frac{n!}{(n-r)!}$$

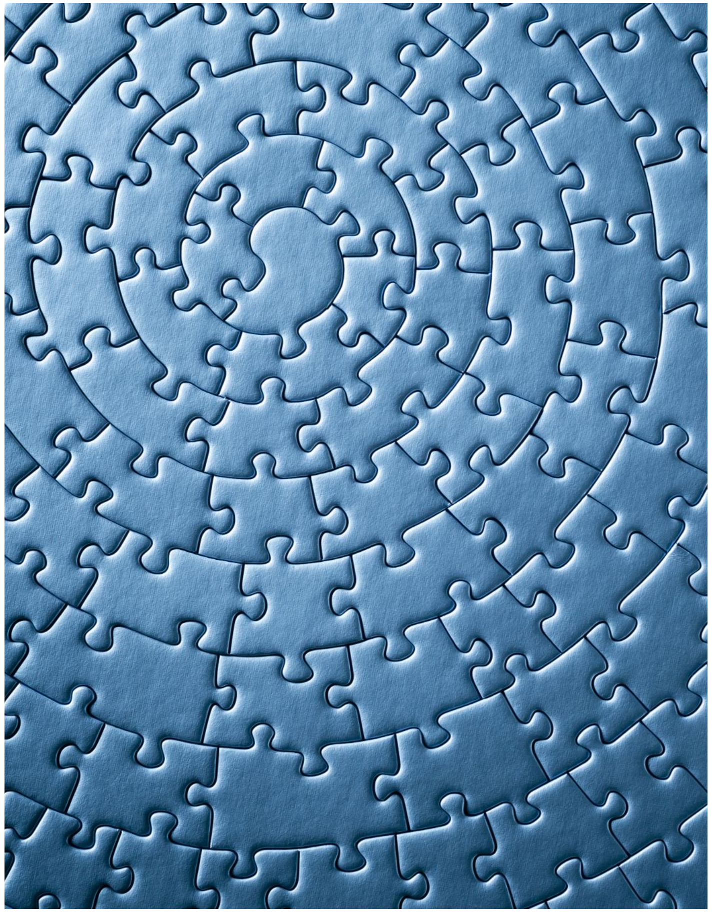
$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Example 1

In how many ways can 3 students be selected and arranged as student union representatives from 5 students?

Here: $n = 5$ and $r = 3$

$$P(5,3) = \frac{5!}{3!} = \frac{120}{2} = 60$$



Special case
(permutations of all objects)

When **all n objects are arranged**, the number of permutations is:

$$P(n,n) = n!$$

Example 1

How many ways can 4 books be arranged on a shelf?

$$4! = 24$$

Example 2

In how many different ways can 4 students be seated in 4 distinct chairs arranged in a row?

Step 1: Identify the counting type

- **Order matters**: Sitting A–B–C–D is different from B–A–C–D.
- **No repetition**: Each student can occupy **only one chair**.

Therefore, this is a problem of **permutations without repetition.**

Step 2: Apply the formula

Here: $n = 4$ (number of students),$r = 4$ (number of positions)

$P(4, 4) = 4!$ = 24 **different seating arrangements** of the 4 students.

Formula

If there are **n distinct objects** and we form an arrangement of **r positions**, allowing repetition, then the number of permutations is:

$$n^r$$

Example 1

How many **4-digit PIN codes** can be formed using the digits 0–9?

$$n = 10\text{digits};\ r = 4\text{positions}$$
$$10^4 = 10{,}000$$

Here, repetition is allowed because digits can appear more than once.

Example 2

In how many different ways can a 3-letter password be formed using the letters $\{A, B, C, D\}$ if letters may be repeated?

Identify the Counting Type

- **Order matters**: ABC is different from BCA, so arrangement matters.
- **Repetition is allowed**: A letter can be used more than once (e.g., AAB, DDD).

Therefore, this is a case of **permutations with repetition**.

Finish up the solution

Combinations (order does not matter)

- A **combination** is a **selection of objects where order is not important**.
- If changing the order does **not** produce a new outcome, then we are dealing with combinations.

$\{A, B, C\}$ and $\{C, B, A\}$ are considered identical.

Combinations without repetition

Motivation

Consider selecting committee members:

- The group {A, B, C} is the same as {C, B, A}
- Order does not matter

Formula

The number of combinations of **n objects taken r at a time** is:

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Example 1

In how many ways can **3 students** be selected from **5 students** to form a committee?

$$C(5,3) = \frac{5!}{3!\,2!} = \frac{120}{12} = 10$$

Example 2

A department has **8 students**. In how many different ways can a **research team of 3 students** be selected?

Identify the Counting Type

- **Order does not matter**: The team {A, B, C} is the same as {C, B, A}.
- **Repetition is not allowed**: A student cannot be selected more than once.

Therefore, this is a **combination without repetition** problem.

Identify Parameters

Number of students: $n = 8$; Number of students selected: $r = 3$

Finish up the solution

Combinations with repetition allowed

Situation

This occurs when:

- Objects can be chosen more than once
- Order does not matter

Formula

The number of combinations of **n distinct objects**, taken **r at a time**, with repetition allowed is:

$$C\ (n + r - 1, r)$$

Example 1

How many ways can **3 sweets** be selected from **5 types of sweets**, if repetition is allowed?

$$C(5+3-1,3)=C(7,3)=35$$

Example 2

A cafeteria offers **5 types of fruits**: Apple, Banana, Orange, Mango, Pineapple. In how many different ways can a student select **3 fruits,** if the same type of fruit can be chosen more than once?

Identify the Counting Type

- **Order does not matter**: Selecting (Apple, Banana, Mango) is the same as (Mango, Banana, Apple).
- **Repetition is allowed**: The student may choose more than one fruit of the same type (e.g., Apple, Apple, Orange).

Therefore, this is a **combination with repetition** problem.

Finish up the solution

Common mistakes to avoid

- Using permutations when **order does not matter**
- Forgetting whether **repetition is allowed**
- Confusing $P(n, r)$ with $C(n, r)$

Summary table

Situation	**Order Matters?**	**Repetition Allowed?**	**Formula**
Permutation	Yes	No	$\frac{n!}{(n-r)!}$
Permutation	Yes	Yes	n^r
Combination	No	No	$\frac{n!}{r!(n-r)!}$
Combination	No	Yes	$C(n+r-1, r)$